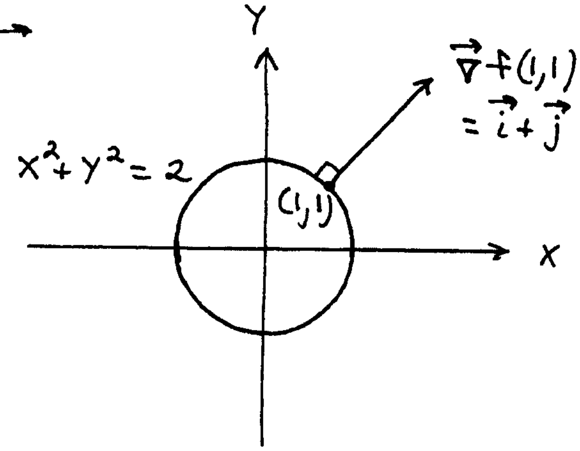
Section 14.5

2.) $f(x,y) = \ln(x^2+y^2)$ and $P=(1,1)$;
$f(1,1) = \ln 2$ then level curve at $(1,1)$
is $\ln 2 = \ln(x^2+y^2) \rightarrow$
$\underline{x^2+y^2=2}$; and
$f_x = \dfrac{2x}{x^2+y^2}$ and
$f_y = \dfrac{2y}{x^2+y^2}$ so
gradient vector is
$\vec{\nabla} f(1,1) = f_x(1,1)\,\vec{i} + f_y(1,1)\,\vec{j}$
$= (1)\,\vec{i} + (1)\,\vec{j} = \vec{i}+\vec{j}$



7) $f(x,y,z) = x^2+y^2-2z^2+z\ln x$
and $P=(1,1,1)$; then
$f_x = 2x + z\cdot\frac{1}{x}$, $f_y = 2y$, $f_z = -4z+\ln x$
so that the gradient vector is
$\vec{\nabla} f(1,1,1) = f_x(1,1,1)\,\vec{i} + f_y(1,1,1)\,\vec{j} + f_z(1,1,1)\,\vec{k}$

$$= 3\vec{i} + 2\vec{j} - 4\vec{k}$$

10) $f(x,y,z) = e^{x+y}\cos z + (y+1)\arcsin x$ and $P = (0, 0, \pi/6)$; then

$$f_x = e^{x+y}\cos z + (y+1)\cdot\frac{1}{\sqrt{1-x^2}} ,$$

$$f_y = e^{x+y}\cos z + \arcsin x ,$$

$f_z = -e^{x+y}\cdot\sin z$ so that the gradient vector is

$$\vec{\nabla} f(0,0,\tfrac{\pi}{6}) = f_x(0,0,\tfrac{\pi}{6})\vec{i} + f_y(0,0,\tfrac{\pi}{6})\vec{j} + f_z(0,0,\tfrac{\pi}{6})\vec{k}$$

$$= \left(\frac{\sqrt{3}}{2}+1\right)\vec{i} + \frac{\sqrt{3}}{2}\vec{j} - \frac{1}{2}\vec{k}$$

12) $f(x,y) = 2x^2 + y^2$, $P = (-1,1)$, and $\vec{A} = 3\vec{i} - 4\vec{j}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{5}(3\vec{i} - 4\vec{j}) = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j} \text{ ; and}$$

$f_x = 4x$, $f_y = 2y$ so gradient vector

$\vec{\nabla} f(-1,1) = f_x(-1,1)\vec{i} + f_y(-1,1)\vec{j}$; then

$$D_{\vec{u}} f(-1,1) = \vec{\nabla} f(-1,1)\cdot\vec{u}$$

$$= (-4\vec{i} + 2\vec{j})\cdot\left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$$

$$= \frac{-12}{5} - \frac{8}{5} = -4$$

15) $f(x,y,z) = xy + yz + xz$, $P = (1,-1,2)$, and

$\vec{A} = 3\vec{i} + 6\vec{j} - 2\vec{k}$; then direction vector
$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{7}(3\vec{i} + 6\vec{j} - 2\vec{k}) = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}$;
and $f_x = y + z$, $f_y = x + z$, $f_z = x + y$ so
gradient vector is

$$\vec{\nabla} f(1,-1,2) = f_x(1,-1,2)\vec{i} + f_y(1,-1,2)\vec{j} + f_z(1,-1,2)\vec{k}$$
$$= (1)\vec{i} + (3)\vec{j} + (0)\vec{k} = \vec{i} + 3\vec{j}$$; then

$$D_{\vec{u}} f(1,-1,2) = \vec{\nabla} f(1,-1,2) \cdot \vec{u}$$
$$= (\vec{i} + 3\vec{j}) \cdot \left(\frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}\right)$$
$$= \frac{3}{7} + \frac{18}{7} = 3 .$$

16) $f(x,y,z) = x^2 + 2y^2 - 3z^2$, $P = (1,1,1)$, and
$\vec{A} = \vec{i} + \vec{j} + \vec{k}$; then direction vector
$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}) = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$;
and $f_x = 2x$, $f_y = 4y$, $f_z = -6z$ so
gradient vector is

$$\vec{\nabla} f(1,1,1) = f_x(1,1,1)\vec{i} + f_y(1,1,1)\vec{j} + f_z(1,1,1)\vec{k}$$
$$= (2)\vec{i} + (4)\vec{j} + (-6)\vec{k} = 2\vec{i} + 4\vec{j} - 6\vec{k}$$;

then $D_{\vec{u}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \vec{u}$
$$= (2\vec{i} + 4\vec{j} - 6\vec{k}) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right)$$
$$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{-6}{\sqrt{3}} = 0 .$$

19) $f(x,y) = x^2 + xy + y^2$ and $P = (-1,1)$;
$f_x = 2x + y$, $f_y = x + 2y$ so gradient vector is

$$\vec{\nabla} f(-1,1) = f_x(-1,1)\,\vec{i} + f_y(-1,1)\,\vec{j}$$
$$= (-1)\,\vec{i} + (1)\,\vec{j} = -\vec{i} + \vec{j}$$;

a.) The direction of maximum <u>increase</u> is in the <u>same</u> direction as $\vec{\nabla} f(-1,1)$, i.e.,

$$\vec{u} = \frac{1}{|\vec{\nabla} f(-1,1)|}\,\vec{\nabla} f(-1,1) = \frac{1}{\sqrt{2}}(-\vec{i} + \vec{j}) = \frac{-1}{\sqrt{2}}\,\vec{i} + \frac{1}{\sqrt{2}}\,\vec{j}$$;

the dir. der. in this direction is

$$D_{\vec{u}} f(-1,1) = |\vec{\nabla} f(-1,1)| = \sqrt{2}$$.

b.) The direction of maximum <u>decrease</u> is in the <u>opposite</u> direction of $\vec{\nabla} f(-1,1)$, i.e., $\vec{u} = \frac{1}{\sqrt{2}}\,\vec{i} - \frac{1}{\sqrt{2}}\,\vec{j}$; the dir. der. in this direction is

$$D_{\vec{u}} f(-1,1) = -|\vec{\nabla} f(-1,1)| = -\sqrt{2}$$

22) $g(x,y,z) = xe^y + z^2$ and $P = (1, \ln 2, 1/2)$;
$g_x = e^y$, $g_y = xe^y$, $g_z = 2z$ so gradient vector is

$$\vec{\nabla} g(1, \ln 2, 1/2) = g_x(1, \ln 2, \tfrac{1}{2})\,\vec{i} + g_y(1, \ln 2, \tfrac{1}{2})\,\vec{j} + g_z(1, \ln 2, \tfrac{1}{2})\,\vec{k}$$
$$= (2)\,\vec{i} + (2)\,\vec{j} + (1)\,\vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}$$

a.) The direction of maximum <u>increase</u> is in the <u>same</u> direction as $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$, i.e., $\vec{u} = \dfrac{1}{|\vec{\nabla} g(1, \ln 2, \frac{1}{2})|} \vec{\nabla} g(1, \ln 2, \frac{1}{2})$

$= \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$;

the dir. der. in this direction is

$D_{\vec{u}}\, g(1, \ln 2, \frac{1}{2}) = |\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = 3$.

b.) The direction of maximum <u>decrease</u> is in the <u>opposite</u> direction of $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$, i.e., $\vec{u} = \frac{-2}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$; the dir. der. in this direction is

$D_{\vec{u}}\, g(1, \ln 2, \frac{1}{2}) = -|\nabla g(1, \ln 2, \frac{1}{2})| = -3$.

31) Let $f(x,y) = xy + y^2$ and $P = (3,2)$.

For what $\vec{u}$ is $D_{\vec{u}} f(3,2) = 0$?

$D_{\vec{u}} f(3,2) = \vec{\nabla} f(3,2) \cdot \vec{u} = 0 \Rightarrow$

$\vec{u}$ is parallel to $\vec{T} = f_y(3,2)\vec{i} - f_x(3,2)\vec{j}$;

$f_x = y$ and $f_y = x + 2y$ so

$\vec{T} = 7\vec{i} - 2\vec{j}$; then

$\vec{u} = \dfrac{1}{|\vec{T}|}\vec{T} = \dfrac{1}{\sqrt{53}}(7\vec{i} - 2\vec{j}) = \dfrac{7}{\sqrt{53}}\vec{i} - \dfrac{2}{\sqrt{53}}\vec{j}$

or $\vec{u} = \dfrac{-1}{|\vec{T}|}\vec{T} = \dfrac{-7}{\sqrt{53}}\vec{i} + \dfrac{2}{\sqrt{53}}\vec{j}$

35) If $\vec{A} = \vec{i} + \vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = 2\sqrt{2} \Rightarrow$

$\left(f_x(1,2)\vec{i} + f_y(1,2)\vec{j}\right) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = 2\sqrt{2} \Rightarrow$

$f_x(1,2) \cdot \frac{1}{\sqrt{2}} + f_y(1,2) \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2} \Rightarrow$

$\boxed{f_x(1,2) + f_y(1,2) = 4}$;

if $\vec{A} = -2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{2}(-2\vec{j}) = -\vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = -3 \Rightarrow$

$\left(f_x(1,2)\vec{i} + f_y(1,2)\vec{j}\right) \cdot (-\vec{j}) = -3 \Rightarrow$

$-f_y(1,2) = -3 \Rightarrow \boxed{f_y(1,2) = 3}$; then

$f_x(1,2) + (3) = 4 \Rightarrow \boxed{f_x(1,2) = 1}$;

now the gradient vector is

$\underline{\underline{\vec{\nabla} f(1,2)}} = f_x(1,2)\vec{i} + f_y(1,2)\vec{j} = \underline{\underline{\vec{i} + 3\vec{j}}}$;

if $\vec{A} = -\vec{i} - 2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{5}}(-\vec{i} - 2\vec{j})$

$\Rightarrow \vec{u} = \frac{-1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}$; then

$$\begin{aligned} D_{\vec{u}} f(1,2) &= \vec{\nabla} f(1,2) \cdot \vec{u} \\ &= (\vec{i} + 3\vec{j}) \cdot \left(\frac{-1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}\right) \\ &= \frac{-1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = \frac{-7}{\sqrt{5}} \end{aligned}$$

Let $P = (a, b, c)$ and $w = f(x, y, z)$.

36) a.) If $\vec{v} = \vec{i} + \vec{j} - \vec{k}$, then

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k}) = \underline{\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}} ;$$

since $D_{\vec{u}} f(a,b,c) = \vec{\nabla} f(a,b,c) \cdot \vec{u} = 2\sqrt{3} \Rightarrow$

$$(f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}\right) = 2\sqrt{3} \Rightarrow$$

$$f_x(a,b,c) \cdot \frac{1}{\sqrt{3}} + f_y(a,b,c) \cdot \frac{1}{\sqrt{3}} - f_z(a,b,c) \cdot \frac{1}{\sqrt{3}} = 2\sqrt{3} \Longrightarrow$$

(1) $$\boxed{f_x(a,b,c) + f_y(a,b,c) + f_z(a,b,c) = 6} ;$$

since $D_{\vec{u}} f(a,b,c)$ is the <u>largest</u> value of a directional derivative, $\vec{u}$ and $\vec{\nabla} f(a,b,c)$ point in the same direction, i.e.,

$$\boxed{\vec{\nabla} f(a,b,c) = k\vec{u}}$$

for some constant k. Then

$$f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k} = \frac{k}{\sqrt{3}}\vec{i} + \frac{k}{\sqrt{3}}\vec{j} - \frac{k}{\sqrt{3}}\vec{k} \Rightarrow$$

$$f_x(a,b,c) = \frac{k}{\sqrt{3}},\ f_y(a,b,c) = \frac{k}{\sqrt{3}},\ f_z(a,b,c) = \frac{-k}{\sqrt{3}} \Longrightarrow$$

(use (1) now.) $$\frac{k}{\sqrt{3}} + \frac{k}{\sqrt{3}} + \frac{-k}{\sqrt{3}} = 6 \Rightarrow$$

$$\frac{k}{\sqrt{3}} = 6 \Rightarrow k = 6\sqrt{3} \Longrightarrow$$

$$\boxed{\vec{\nabla} f(a,b,c) = 6\vec{i} + 6\vec{j} - 6\vec{k}} .$$