

Section 14.7

1.) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4 \Rightarrow$

$$f_x = 2x + y + 3 = 0 \Rightarrow \boxed{y = -2x - 3} ;$$

$$f_y = x + 2y - 3 = 0 \Rightarrow \boxed{x = 3 - 2y} ;$$

substitute \Rightarrow

$$y = -2x - 3 = -2(3 - 2y) - 3 \Rightarrow$$

$$y = -6 + 4y - 3 \Rightarrow 9 = 3y \Rightarrow$$

$y = 3 \Rightarrow x = -3$ so $\boxed{(-3, 3)}$ is critical

point ; $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 1$;
then

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (2)(2) - (1)^2 = 3 > 0$$

and $f_{xx} = 2 > 0$, so $(-3, 3)$

determines a minimum value
of $f(-3, 3) = -5$.

3) $f(x,y) = x^2 + xy + 3x + 2y + 5 \Rightarrow$

$$f_x = 2x + y + 3 = 0 \Rightarrow \boxed{y = -2x - 3} ;$$

$$f_y = x + 2 = 0 \Rightarrow \boxed{x = -2} ;$$

substitute \Rightarrow

$$y = -2x - 3 = -2(-2) - 3 = 1 \Rightarrow$$

$\boxed{(-2, 1)}$ is critical point ;

$f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = 1$; then

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (2)(0) - (1)^2 = -1 < 0, \text{ so}$$

$(-2, 1)$ determines a saddle point.

$$15) f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy \Rightarrow$$

$$f_x = 12x - 6x^2 + 6y = 6(2x - x^2 + y) = 0 \Rightarrow$$

$$2x - x^2 + y = 0 \Rightarrow \boxed{y = x^2 - 2x};$$

$$f_y = 6y + 6x = 6(y + x) = 0 \Rightarrow y + x = 0 \Rightarrow$$

$$\boxed{y = -x}; \text{ substitute } \Rightarrow$$

$$y = x^2 - 2x \Rightarrow -x = x^2 - 2x \Rightarrow$$

$$0 = x^2 - x = x(x-1) \Rightarrow \underline{x=0} \text{ or } \underline{x=1};$$

if $x=0$, then $y=0$ so $\boxed{(0,0)}$ is critical point; if $x=1$, then $y=-1$, so $\boxed{(1,-1)}$ is critical point;

$$f_{xx} = 12 - 12x, f_{yy} = 6, f_{xy} = 6;$$

check (0,0): $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$$= (12)(6) - (6)^2 = 36 > 0, \text{ and}$$

$f_{xx} = 12 > 0$ so $(0,0)$ determines a minimum value of $f(0,0) = 0$.

check (1,-1): $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$$= (0)(6) - (6)^2 = -36 < 0, \text{ so}$$

$(1,-1)$ determines a saddle point.

$$16) f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8 \Rightarrow$$

$$f_x = 3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow$$

$$\boxed{x=0} \text{ or } \boxed{x=-2};$$

$$f_y = 3y^2 - 6y = 3y(y-2) \Rightarrow$$

$$\boxed{y=0} \text{ or } \boxed{y=2}; \text{ then}$$

critical points are

$$(0,0), (0,2), (-2,0), (-2,2);$$
$$f_{xx} = 6x+6, f_{yy} = 6y-6, f_{xy} = 0;$$

check $(0,0)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
 $= (6)(-6) - (0)^2 = -36 < 0$, so
 $(0,0)$ determines a saddle point;

check $(0,2)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
 $= (6)(6) - (0)^2 = 36 > 0$, and
 $f_{xx} = 6 > 0$ so $(0,2)$ determines a
minimum value of $f(0,2) = -12$.

check $(-2,0)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
 $= (-6)(-6) - (0)^2 = 36 > 0$, and
 $f_{xx} = -6 < 0$ so $(-2,0)$ determines a
maximum value of $f(-2,0) = -4$;

check $(-2,2)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
 $= (-6)(6) - (0)^2 = -36 < 0$, so
 $(-2,2)$ determines a saddle point.

19) $f(x,y) = 4xy - x^4 - y^4 \Rightarrow$
 $f_x = 4y - 4x^3 = 4(y - x^3) = 0 \Rightarrow \boxed{y = x^3}$;
 $f_y = 4x - 4y^3 = 4(x - y^3) = 0 \Rightarrow \boxed{x = y^3}$;
substitute \Rightarrow
 $y = x^3 = (y^3)^3 = y^9 \Rightarrow$

$$\begin{aligned}
 0 &= Y^9 - Y = Y(Y^8 - 1) = Y(Y^4 - 1)(Y^4 + 1) \\
 &= Y(Y^2 - 1)(Y^2 + 1)(Y^4 + 1) \\
 &= Y(Y - 1)(Y + 1)(Y^2 + 1)(Y^4 + 1) \\
 &\quad \downarrow \qquad \qquad \qquad \rightarrow \qquad \qquad \qquad \rightarrow \qquad \qquad \qquad \rightarrow \\
 &Y = 0 \text{ or } Y = 1 \text{ or } Y = -1 ;
 \end{aligned}$$

if $Y = 0$, then $X = 0$ so $(0, 0)$ is critical point; if $Y = 1$, then $X = 1$ so $(1, 1)$ is critical point; if $Y = -1$, then $X = -1$ so $(-1, -1)$ is critical point;

$$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4 ;$$

check $(0, 0)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$$= (0)(0) - (4)^2 = -16 < 0, \text{ so}$$

$(0, 0)$ determines a saddle point;

check $(1, 1)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$$= (-12)(-12) - (4)^2 = 128 > 0, \text{ and}$$

$f_{xx} = -12 < 0$ so $(1, 1)$ determines a maximum value of $f(1, 1) = 2$;

check $(-1, -1)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$$= (-12)(-12) - (4)^2 = 128 > 0, \text{ and}$$

$f_{xx} = -12 < 0$ so $(-1, -1)$ determines a maximum value of $f(-1, -1) = 2$.

22) $f(x, y) = \frac{1}{x} + xy + \frac{1}{y} \Rightarrow$

$$f_x = \frac{-1}{x^2} + y = 0 \Rightarrow \boxed{y = \frac{1}{x^2}} ;$$

$$f_y = x - \frac{1}{y^2} = 0 \Rightarrow \boxed{x = \frac{1}{y^2}} ;$$

substitute \Rightarrow

$$y = \frac{1}{x^2} = \frac{1}{\left(\frac{1}{y^2}\right)^2} = y^4 \Rightarrow$$

$$y = y^4 \Rightarrow 0 = y^4 - y = y(y^3 - 1) \Rightarrow$$

$y = 0$ or $y = 1$; if $y = 0$, then
 $x = \frac{1}{y^2}$ (impossible!) ; if $y = 1$,

then $x = 1$ so $\boxed{(1,1)}$ is critical

point ; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$,

$f_{xy} = 1$; then

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (2)(2) - (1)^2 = 3 > 0, \text{ and}$$

$f_{xx} = 2 > 0$, so $(1,1)$ determines
a minimum value of

$$f(1,1) = 3 .$$