1.) Find extrema for $f(x,y) = xy$

s.t. $x^2 + 2y^2 = 1$:

Let $F(x,y,\lambda) = xy - \lambda(x^2 + 2y^2 - 1)$ $\longrightarrow$

$\left. \begin{array}{l} F_x = y - 2\lambda x = 0 \\ F_y = x - 4\lambda y = 0 \end{array} \right\} \quad \left. \begin{array}{l} \lambda = \frac{y}{2x} \\ \lambda = \frac{x}{4y} \end{array} \right\} \quad \frac{y}{2x} = \frac{x}{4y} \longrightarrow \boxed{2y^2 = x^2}$

$F_\lambda = -x^2 - 2y^2 + 1 = 0$ (sub.) $\longleftarrow$

$\longrightarrow -x^2 - x^2 + 1 = 0 \longrightarrow 2x^2 = 1 \longrightarrow$

$x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$, if $x = \frac{1}{\sqrt{2}}$, then

$y = \pm \frac{1}{2}$ so $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$ are

critical points; if $x = \frac{-1}{\sqrt{2}}$, then

$y = \pm \frac{1}{2}$ so $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$ are

critical points :

crit. pts.	values of $f(x,y)$	
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$	$\frac{1}{2\sqrt{2}}$	MAX
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$	$\frac{-1}{2\sqrt{2}}$	MIN
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$	$\frac{-1}{2\sqrt{2}}$	MIN
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$	$\frac{1}{2\sqrt{2}}$	MAX

3.) Maximize $f(x,y) = 49 - x^2 - y^2$ s.t.

$x + 3y = 10$:

Let $F(x,y,\lambda) = (49 - x^2 - y^2) - \lambda(x + 3y - 10)$

$\longrightarrow \left. \begin{array}{l} F_x = -2x - \lambda = 0 \\ F_y = -2y - 3\lambda = 0 \end{array} \right\} \quad \left. \begin{array}{l} \lambda = -2x \\ \lambda = -\frac{2}{3}y \end{array} \right\} \quad -2x = -\frac{2}{3}y \longrightarrow \boxed{y = 3x}$

$F_\lambda = -x - 3y + 10 = 0$ (sub.) $\longleftarrow$
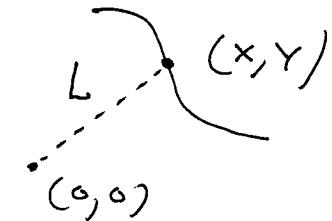
$\longrightarrow -x - 3(3x) + 10 = 0 \longrightarrow$

$$-10X + 10 = 0 \rightarrow X = 1 \text{ and } Y = 3 \text{ so}$$

$\boxed{(1,3)}$ is critical point with maximum value $f(1,3) = 39$.

8.) Distance
$$L = \sqrt{(X-0)^2 + (Y-0)^2} \rightarrow$$
$$L^2 = X^2 + Y^2 \quad;$$

Minimize and maximize
$$f(X,Y) = X^2 + Y^2 \quad \text{s.t.} \quad X^2 + XY + Y^2 = 1 \quad:$$
Let $F(X,Y,\lambda) = (X^2 + Y^2) - \lambda(X^2 + XY + Y^2 - 1) \rightarrow$

$$F_X = 2X - \lambda(2X+Y) = 0 \rightarrow \lambda = \frac{2X}{2X+Y}$$

$$F_Y = 2Y - \lambda(X+2Y) = 0 \rightarrow \lambda = \frac{2Y}{X+2Y}$$

$$\left. \begin{array}{c} \\ \\ \end{array} \right\} \rightarrow$$

$$\frac{2X}{2X+Y} = \frac{2Y}{X+2Y} \rightarrow 2X^2 + 4XY = 4XY + 2Y^2 \rightarrow$$
$$\boxed{Y^2 = X^2} \rightarrow \underline{\underline{Y=X}} \text{ or } \underline{\underline{Y=-X}} \;;$$

$$F_\lambda = -X^2 - XY - Y^2 + 1 = 0 \qquad (\text{sub.}) \; 5 \nearrow$$

If $Y = X$, then $-X^2 - X^2 - X^2 + 1 = 0 \rightarrow$
$$3X^2 = 1 \rightarrow X = \pm 1/\sqrt{3} \rightarrow \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and}$$
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ are critical points ;

If $Y = -X$, then $-X^2 + X^2 - X^2 + 1 = 0 \rightarrow$
$$X^2 = 1 \rightarrow X = \pm 1 \rightarrow \underline{(1,-1)} \text{ and } \underline{(-1,1)} \text{ are}$$
critical points ; then

$$\underline{\text{crit. pts.}} \qquad \underline{\text{values of } f(x,y)}$$

$$\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) \qquad \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3} \qquad MIN$$

$$\left(\tfrac{-1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}\right) \qquad \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3} \qquad MIN$$

$$(1,-1) \qquad 1+1 = 2 \qquad MAX$$

$$(-1,1) \qquad 1+1 = 2 \qquad MAX$$

So min. distance is $\sqrt{\tfrac{2}{3}}$; max. distance is $\sqrt{2}$.

14.) maximize and minimize

$$f(x,y) = 3x - y + 6 \quad \text{s.t.} \quad x^2 + y^2 = 4 :$$

Let $F(x,y,\lambda) = (3x - y + 6) - \lambda(x^2 + y^2 - 4) \longrightarrow$

$$F_x = 3 - \lambda(2x) = 0 \longrightarrow \lambda = \tfrac{3}{2x} \left.\right\} \tfrac{3}{2x} = \tfrac{-1}{2y} \longrightarrow$$

$$F_y = -1 - \lambda(2y) = 0 \longrightarrow \lambda = \tfrac{-1}{2y} \left.\right\} \boxed{y = \tfrac{-1}{3}x} \; ;$$

$$F_\lambda = -x^2 - y^2 + 4 = 0 \qquad (\text{sub.}) \hookleftarrow$$

$$\longrightarrow -x^2 - \left(\tfrac{-1}{3}x\right)^2 + 4 = 0 \longrightarrow 4 = \tfrac{10}{9}x^2 \longrightarrow$$

$$x^2 = \tfrac{18}{5} \longrightarrow x = \pm\sqrt{\tfrac{18}{5}} = \pm 3\sqrt{\tfrac{2}{5}} \; ;$$

if $x = 3 \cdot \sqrt{\tfrac{2}{5}}$, then $y = -\sqrt{\tfrac{2}{5}}$ and $\underline{\left(3\sqrt{\tfrac{2}{5}}, -\sqrt{\tfrac{2}{5}}\right)}$ is a critical point ;

if $x = -3\sqrt{\tfrac{2}{5}}$, then $y = \sqrt{\tfrac{2}{5}}$ and $\underline{\left(-3\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{2}{5}}\right)}$ is a critical point ; then

$$\underline{\text{crit. pts.}} \qquad \underline{\text{values of } f(x,y)}$$
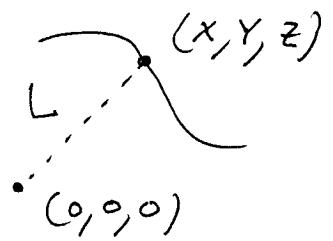
$$\left(3\sqrt{\tfrac{2}{5}}, -\sqrt{\tfrac{2}{5}}\right) \qquad 9\sqrt{\tfrac{2}{5}} + \sqrt{\tfrac{2}{5}} + 6 = 10\sqrt{\tfrac{2}{5}} + 6 \qquad MAX$$

$$\left(-3\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{2}{5}}\right) \qquad -9\sqrt{\tfrac{2}{5}} - \sqrt{\tfrac{2}{5}} + 6 = -10\sqrt{\tfrac{2}{5}} + 6 \qquad MIN$$

21.) Distance

$$L = \sqrt{(x-0)^2 + (Y-0)^2 + (z-0)^2}$$

$$\rightarrow \quad L^2 = x^2 + Y^2 + z^2 \; ;$$

minimize $f(x,Y,z) = x^2 + Y^2 + z^2$

s.t. $z^2 = XY + 4$:

Let $F(x,Y,z,\lambda) = (x^2 + Y^2 + z^2) - \lambda(z^2 - XY - 4) \rightarrow$

$F_x = 2x - \lambda(-Y) = 0 \rightarrow \lambda = \frac{-2x}{Y}$ $\Big\}$ $1 = \frac{-2x}{Y} \rightarrow Y = -2x$,

$F_Y = 2Y - \lambda(-x) = 0 \rightarrow \lambda = \frac{-2Y}{x}$ $\Big\}$ $1 = \frac{-2Y}{x} \rightarrow Y = \frac{-1}{2}x$

$F_z = 2z - \lambda(2z) = 0 \rightarrow \lambda = 1$ $\rightarrow -2x = -\frac{1}{2}x$

$\rightarrow 4x = x \rightarrow 3x = 0 \rightarrow x = 0, \; Y = 0 \; ;$

$F_\lambda = -z^2 + XY + 4 = 0 \rightarrow -z^2 + (0)(0) + 4 = 0$

$\rightarrow z^2 = 4 \rightarrow z = \pm 2$ so $\boxed{(0,0,2)}$

and $\boxed{(0,0,-2)}$ are critical points ;
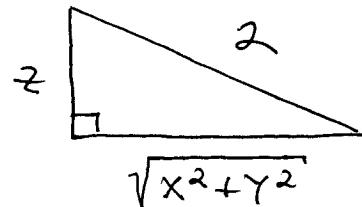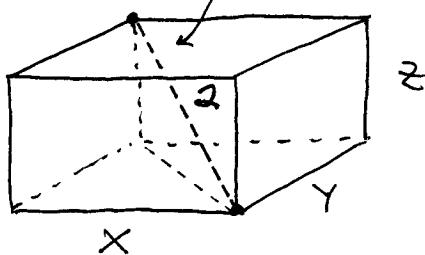
| crit. pts.	values of $f(x,Y)$	
$(0,0,2)$	$0 + 0 + 4 = 4$	MIN
$(0,0,-2)$	$0 + 0 + 4 = 4$	MIN

So minimum distance is $\sqrt{4} = 2$.

       diagonal of box is   2

27.)

then $x^2 + Y^2 + z^2 = 4 \; ;$

Maximize volume $V = XYZ$

s.t. $\quad X^2 + Y^2 + Z^2 = 4$: $(X>0, Y>0, Z>0)$

Let $F(X,Y,Z,\lambda) = XYZ - \lambda(X^2 + Y^2 + Z^2 - 4) \rightarrow$

$F_X = YZ - \lambda(2X) = 0 \rightarrow \lambda = \dfrac{YZ}{2X}$

$F_Y = XZ - \lambda(2Y) = 0 \rightarrow \lambda = \dfrac{XZ}{2Y}$ $\Big\}$ (pair them) $\rightarrow$

$F_Z = XY - \lambda(2Z) = 0 \rightarrow \lambda = \dfrac{XY}{2Z}$

$\qquad \dfrac{YZ}{2X} = \dfrac{XZ}{2Y} \rightarrow \boxed{X^2 = Y^2}$

$\qquad \dfrac{XZ}{2Y} = \dfrac{XY}{2Z} \rightarrow \boxed{Z^2 = Y^2}$ $\Big\}$ (sub)

$F_\lambda = -X^2 - Y^2 - Z^2 + 4 = 0$

$\qquad \rightarrow -(Y^2) - Y^2 - (Y^2) + 4 = 0$

$\qquad \rightarrow 3Y^2 = 4 \rightarrow Y = \pm \dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$ (ONLY +!)

if $Y = \dfrac{2}{\sqrt{3}}$, then $X = \dfrac{2}{\sqrt{3}}$ and $Z = \dfrac{2}{\sqrt{3}}$

so critical point is $\boxed{\left(\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)}$ and

maximum volume is

$\qquad V = \left(\dfrac{2}{\sqrt{3}}\right)^3 = \dfrac{8}{3\sqrt{3}}$

30.) Maximize and minimize $T = 400 XYZ^2$

s.t. $\quad X^2 + Y^2 + Z^2 = 1$:

Let $F(X,Y,Z,\lambda) = 400 XYZ^2 - \lambda(X^2 + Y^2 + Z^2 - 1) \rightarrow$

$F_X = 400 YZ^2 - \lambda(2X) = 0 \rightarrow \lambda = \dfrac{400 YZ^2}{2X}$

$$\rightarrow \quad \lambda = \frac{200\,Y\,Z^2}{X} \quad ;$$

$$F_Y = 400X\,Z^2 - \lambda\,(2Y) = 0 \rightarrow \lambda = \frac{400\,X\,Z^2}{2Y}$$

$$\rightarrow \quad \lambda = \frac{200\,X\,Z^2}{Y} \quad ;$$

$$F_Z = 800XY Z - \lambda\,(2Z) = 0 \rightarrow \lambda = \frac{800XY Z}{2Z}$$

$$\rightarrow \quad \lambda = \frac{400XY Z}{Z} = 400XY \quad ;$$

(pair λ's) $\rightarrow$

$$\frac{200\,Y\,Z^2}{X} = \frac{200\,X\,Z^2}{Y} \quad \rightarrow \quad Y^2 Z^2 = X^2 Z^2 \left.\begin{array}{}\\\\\end{array}\right\} \rightarrow$$

$$\frac{200\,X\,Z^2}{Y} = 400XY \quad \rightarrow \quad X Z^2 = 2XY^2$$

$$Y^2 Z^2 - X^2 Z^2 = 0 \rightarrow (Y^2 - X^2)\,Z^2 = 0 \left.\begin{array}{}\\\\\end{array}\right\} \rightarrow$$

$$X Z^2 - 2X Y^2 = 0 \rightarrow X(Z^2 - Y^2) = 0$$

$(\ast)$
$$\boxed{\begin{array}{l} Y^2 = X^2 \text{ or } Z = 0 \\ Z^2 = Y^2 \text{ or } X = 0 \end{array}} \left.\begin{array}{}\\\\\end{array}\right\} \quad \text{(sub)} \searrow$$

$$F_\lambda = \boxed{-X^2 - Y^2 - Z^2 + 1 = 0} \quad ; \quad \text{then}$$

<u>case 1:</u>   If $Y^2 = X^2$ and $Z^2 = Y^2$, then

$$-(Y^2) - Y^2 - (Y^2) + 1 = 0 \rightarrow 3Y^2 = 1 \rightarrow$$

$$Y = \pm \tfrac{1}{\sqrt{3}} \quad ; \quad \text{if } Y = \tfrac{1}{\sqrt{3}}, \text{ then } X = \pm\tfrac{1}{\sqrt{3}}$$

and $Z = \pm \tfrac{1}{\sqrt{3}}$ and critical points

are

$$\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}\right), \left(\tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}\right) \, ;$$

if $Y = \frac{-1}{\sqrt{3}}$, then $x = \frac{\pm 1}{\sqrt{3}}$ and $z = \frac{\pm 1}{\sqrt{3}}$
and critical points are
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$;

case 2: If $Y^2 = X^2$ and $x = 0$, then $Y = 0$ and
$0 - 0 - z^2 + 1 = 0 \rightarrow z^2 = 1 \rightarrow z = \pm 1$
so critical points are $(0,0,1), (0,0,-1)$;

case 3: If $z = 0$ and $z^2 = Y^2$, then $Y = 0$
and $-x^2 - 0 - 0 + 1 = 0 \rightarrow x^2 = 1 \rightarrow x = \pm 1$
so critical points are $(1,0,0), (-1,0,0)$;

case 4: If $z = 0$ and $x = 0$, then
$0 - Y^2 - 0 + 1 = 0 \rightarrow Y^2 = 1 \rightarrow Y = \pm 1$
so critical points are $(0,1,0), (0,-1,0)$;
then

critical pts.	values of $f(x,y)$	
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{400}{9}\,^\circ C$	MAX
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{400}{9}\,^\circ C$	MAX
$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$-\frac{400}{9}\,^\circ C$	MIN
$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$-\frac{400}{9}\,^\circ C$	MIN
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$-\frac{400}{9}\,^\circ C$	MIN
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$-\frac{400}{9}\,^\circ C$	MIN
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{400}{9}\,^\circ C$	MAX
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{400}{9}\,^\circ C$	MAX
$(0,0,1)$	$0\,^\circ C$	
$(0,0,-1)$	$0\,^\circ C$	
$(1,0,0)$	$0\,^\circ C$	
$(-1,0,0)$	$0\,^\circ C$	
$(0,1,0)$	$0\,^\circ C$	
$(0,-1,0)$	$0\,^\circ C$	

33.) Maximize $f(x,y,z) = x^2 + 2y - z^2$
 s.t. $2x - y = 0$ and $y + z = 0$:

Let $F(x,y,z,\lambda,\mu) = (x^2 + 2y - z^2)$
$\qquad - \lambda(2x - y) - \mu(y + z) \longrightarrow$

$F_x = 2x - \lambda(2) - \mu(0) = 2x - 2\lambda = 0 \to \lambda = x$
$F_y = 2 - \lambda(-1) - \mu(1) = 2 + \lambda - \mu = 0 \quad \longleftarrow$ (sub.)
$F_z = -2z - \lambda(0) - \mu(1) = -2z - \mu = 0 \to \mu = -2z$
$\qquad \to 2 + x - (-2z) = 0 \longrightarrow \boxed{x + 2z = -2}$;

$F_\lambda = -2x + y = 0 \longrightarrow \boxed{2x - y = 0}$;
$F_\mu = -y - z = 0 \longrightarrow \boxed{y + z = 0}$; then solve

$\left. \begin{array}{l} x \;\;+ 2z = -2 \\ 2x - y \qquad = 0 \\ \;\;\; y + z = 0 \end{array} \right\}$ $\left. \begin{array}{l} x + 2z = -2 \\ 2x + z = 0 \end{array} \right\} \longrightarrow$

$-3z = 4 \longrightarrow z = -4/3$, $y = 4/3,\; x = 2/3$ so
critical point is $\boxed{(2/3, 4/3, -4/3)}$ and
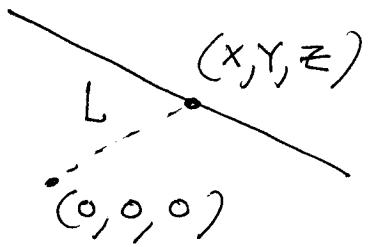
maximum value is
$f(\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{-4}{3}) = \tfrac{4}{9} + \tfrac{24}{9} - \tfrac{16}{9} = \tfrac{12}{9} = \boxed{\tfrac{4}{3}}$.

35.) Distance
$\qquad L = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$\longrightarrow L^2 = x^2 + y^2 + z^2$;

minimize $\quad f(x,y,z) = x^2 + y^2 + z^2$

s.t. $\quad y + 2z = 12$ and $x + y = 6$:

Let $F(x,y,z,\lambda,\mu) = (x^2 + y^2 + z^2)$

$\qquad -\lambda(y + 2z - 12) - \mu(x + y - 6) \longrightarrow$

$F_x = 2x - \lambda(0) - \mu(1) = 2x - \mu = 0 \longrightarrow \mu = 2x$

$F_y = 2y - \lambda(1) - \mu(1) = 2y - \lambda - \mu = 0 \quad \longleftarrow$ (sub.)

$F_z = 2z - \lambda(2) - \mu(0) = 2z - 2\lambda = 0 \longrightarrow \lambda = z$

$\qquad \longrightarrow 2y - z - 2x = 0 \longrightarrow \boxed{2x - 2y + z = 0}$;

$F_\lambda = -y - 2z + 12 = 0 \quad \longrightarrow \boxed{y + 2z = 12}$;

$F_\mu = -x - y + 6 = 0 \quad \longrightarrow \boxed{x + y = 6}$; solve

$\left. \begin{array}{l} x + y = 6 \\ \quad y + 2z = 12 \\ 2x - 2y + z = 0 \end{array} \right\} \quad \left. \begin{array}{l} x - 2z = -6 \\ 2x + 5z = 24 \end{array} \right\} \quad 9z = 36 \longrightarrow$

$z = 4, \quad x = 2, \quad y = 4 \quad$ so critical point

is $\boxed{(2, 4, 4)} \quad$ and minimum

function value is $f(2,4,4) = 4 + 16 + 16 = 36$

so minimum distance is $\sqrt{36} = \boxed{6}$.