Section 10.3

13) $\sum_{n=1}^{\infty} \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$= \frac{1}{1+0} = 1 \neq 0$, so series <u>diverges</u> by <u>nth-term</u> test.

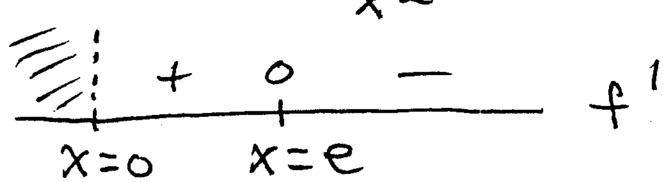
15) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$; series <u>diverges</u> by <u>p-series</u> test since $p = \frac{1}{2} \leq 1$.

16) $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$; series <u>converges</u> by <u>p-series</u> test since $p = \frac{3}{2} > 1$.

17) $\sum_{n=1}^{\infty} \frac{-1}{8^n} = -\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$; series <u>converges</u> by <u>geometric</u> series test since $r = \frac{1}{8}$ and $-1 < r < 1$.

19) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$; Let $f(x) = \frac{\ln x}{x} \xrightarrow{D}$

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x}$, so



f is +, $\downarrow$ for $x \geq 3$,

and continuous for $x \geq 3$; then

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{A\to\infty} \int_3^A \frac{\ln x}{x}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^A = \lim_{A\to\infty} \left(\frac{1}{2}(\ln A)^2 - \frac{1}{2}(\ln 3)^2\right)$$

$= \infty$, so series <u>diverges</u> by the <u>integral test</u>.

22) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$; $\displaystyle\lim_{n\to\infty} \frac{5^n}{4^n+1} = \lim_{n\to\infty} \frac{(5/4)^n}{1+\frac{1}{4^n}}$

$= \dfrac{\infty}{1+0} = \infty \neq 0$, so series <u>diverges</u> by <u>nth-term</u> test.

24) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$; Let $f(x) = \dfrac{1}{2x-1}$, then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{2x-1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln|2x-1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln|2A-1| - \frac{1}{2}\overset{0}{\cancel{\ln 1}}\right) = \infty,$$

so series <u>diverges</u> by <u>integral</u> test.

26) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$; Let $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$$

$$= \lim_{A\to\infty} 2\ln|\sqrt{x}+1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(2\ln|\sqrt{A}+1| - 2\ln 2\right) = \infty,$$

so series <u>diverges</u> by <u>integral test</u>

30) $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{\ln 3}\right)^n$ <u>converges</u>

by <u>geometric</u> series test since $r = \frac{1}{\ln 3}$ and $-1 < r < 1$.

32) $\sum_{n=1}^{\infty} \frac{1}{n(1+(\ln n)^2)}$; Let $f(x) = \frac{1}{x(1+(\ln x)^2)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus, $\int_1^{\infty} \frac{1}{x(1+(\ln x)^2)}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x(1+(\ln x)^2)}\,dx$

$$= \lim_{A\to\infty} \arctan(\ln x) \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\arctan(\ln A) - \arctan(\overset{0}{\cancel{\ln 1}})\right)$$

$$= \arctan(\infty) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

so series <u>converges</u> by <u>integral</u> test.

33) $\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$; $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$= \lim_{w\to 0} \frac{\sin w}{w} = 1 \neq 0$, so series

<u>diverges</u> by <u>nth-term</u> test

35) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$; Let $f(x) = \frac{e^x}{1+e^{2x}} \xrightarrow{D}$

$f'(x) = \frac{(1+e^{2x})\cdot e^x - e^x \cdot 2e^{2x}}{(1+e^{2x})^2}$

$= \frac{1-e^{3x}}{(1+e^{2x})^2}$; $\quad + \quad 0 \quad - \quad f'$ at $x=0$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$;

thus, $\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{A\to\infty} \int_1^A \frac{e^x}{1+(e^x)^2}\,dx$

$= \lim_{A\to\infty} \arctan(e^x)\Big|_1^A$

$= \lim_{A\to\infty} \left(\arctan(e^A) - \arctan(e)\right)$

$= \arctan(\infty) - \arctan(e)$

$= \frac{\pi}{2} - \arctan(e)$, so series

converges.

38) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$; Let $f(x) = \dfrac{x}{x^2+1} \xrightarrow{D}$

$$f'(x) = \frac{(x^2+1)\cdot(1) - x\cdot(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \ ;$$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

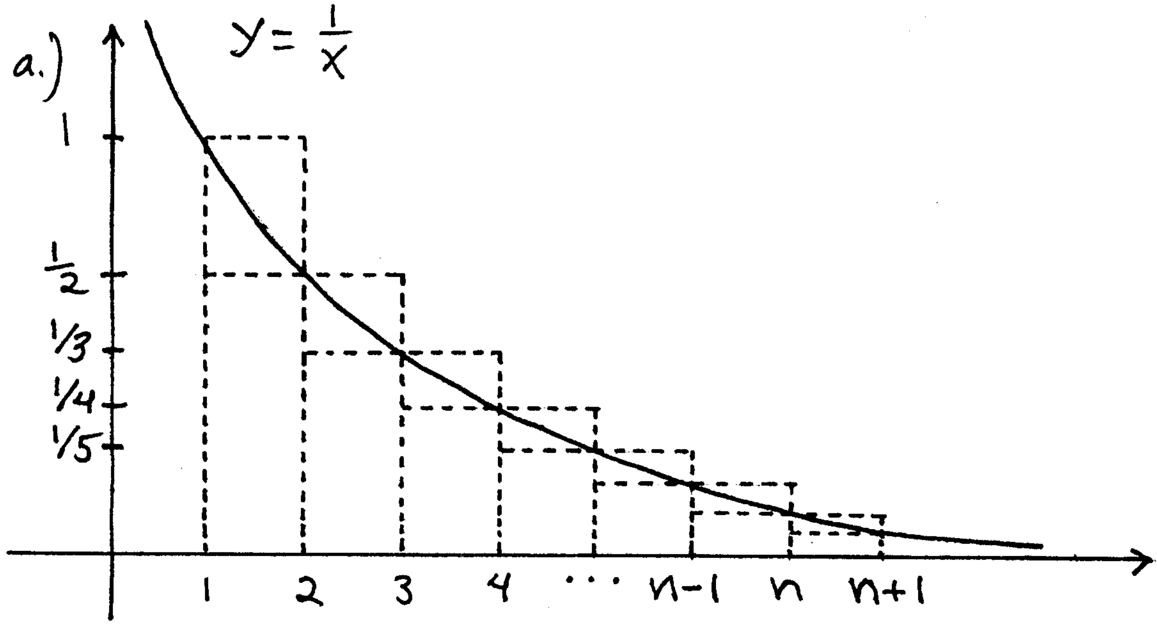
$$\begin{array}{ccc} + & 0 & - \\ \hline & x=1 & \end{array} \quad f'$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{A\to\infty} \int_1^A \frac{x}{x^2+1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln(A^2+1) - \frac{1}{2}\ln 2\right) = \infty \ ,$$ so series <u>diverges</u> by <u>integral</u> test.

43) a.)



Consider the integral $\displaystyle\int_1^{n+1} \frac{1}{x}\,dx$

and rectangles above the graph of $y=\frac{1}{x}$ on the interval $[1, n+1]$. Comparing areas results in

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln x \Big|_1^{n+1} = \ln(n+1) - \cancel{\ln 1}^{\,0}$$

$$\le \left(\frac{1}{1}\right)(1) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1), \text{ i.e.,}$$

(I) $\boxed{\ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$;

Consider the integral $\int_1^n \frac{1}{x}\,dx$ and rectangles below the graph of $y=\frac{1}{x}$ on the interval $[1, n]$. Comparing areas results in

$$\left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{4}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1) \le \int_1^n \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^n = \ln n - \cancel{\ln 1}^{\,0}, \text{ i.e.}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le \ln n \rightarrow$$

(II) $\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + \ln n}$.

b.) $(13{,}000{,}000{,}000 \text{ yrs.})$

$$\times \left(\frac{365 \text{ days}}{\text{yr.}}\right) \times \left(\frac{24 \text{ hrs.}}{\text{day}}\right)$$

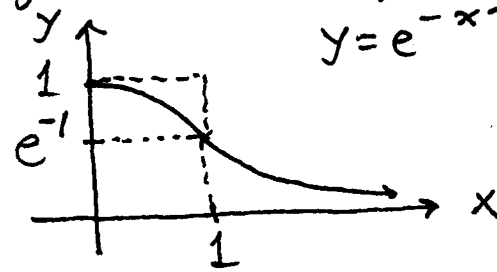
$$\times \left(\frac{60 \text{ min.}}{\text{hr.}}\right) \times \left(\frac{60 \text{ sec.}}{\text{min.}}\right) \approx 4.09968 \times 10^{17} = A \text{ ;}$$

then by (I) and (II)

$$\ln(A+1) \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{A} \le 1+\ln A \rightarrow$$

$$40.5548 \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{A} \le 41.5548 .$$

58) $\sum_{n=0}^{\infty} e^{-n^2}$; Let $f(x)=e^{-x^2}$, then f is $+$, $\downarrow$, and continuous for $x \ge 0$; then



$$\int_0^{\infty} e^{-x^2} = \int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

$$\le (1)(1) + \int_1^{\infty} e^{-x}\,dx$$

$$= 1 + \lim_{A\to\infty} \int_1^A e^{-x}\,dx$$

$$= 1 + \lim_{A\to\infty} -e^{-x}\Big|_1^A$$

$$= 1 + \lim_{A\to\infty} \left(\frac{-1}{e^A} - \frac{-1}{e}\right)$$

$$= 1 + \left(0+\frac{1}{e}\right) = 1+\frac{1}{e} \text{ ; so}$$

$\int_0^{\infty} e^{-x^2}\,dx$ converges.

Problems Using (*) and (*)(*)

1.) a.) $\int_1^{11} \frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\int_1^{10}\frac{1}{x}\,dx \rightarrow$

$\ln x \Big|_1^{11} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln x\Big|_1^{10} \rightarrow$

$\ln 11 - \cancel{\ln 1}^{\,0} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln 10 - \cancel{\ln 1}^{\,0} \rightarrow$

$2.398 \approx \ln 11 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln 10 \approx 3.303$

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b.) $\int_1^{1001}\frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} < 1+\int_1^{1000}\frac{1}{x}\,dx \rightarrow$

$\int_1^{1001}\frac{1}{x}\,dx = \ln x\Big|_1^{1001} = \ln 1001 - \cancel{\ln 1}^{\,0} = \ln 1001$ and

$\int_1^{1000}\frac{1}{x}\,dx = \ln x\Big|_1^{1000} = \ln 1000 - \cancel{\ln 1}^{\,0} = \ln 1000$ so

$6.909 \approx \ln 1001 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} < 1+\ln 1000 \approx 7.908$

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c.) $\int_1^{1,000,001}\frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1,000,000} < 1+\int_1^{1,000,000}\frac{1}{x}\,dx \rightarrow$

$13.816 \approx \ln 1,000,001 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1,000,000} < 1+\ln 1,000,000 \approx 14.816$

2.) a.) $S_{10} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2} \approx 1.550$

---

b.) $\int_{11}^{\infty}\frac{1}{x^2}\,dx < \frac{1}{11^2}+\frac{1}{12^2}+\cdots < \int_{10}^{\infty}\frac{1}{x^2}\,dx \rightarrow$

$\lim_{A\to\infty}\int_{11}^{A}\frac{1}{x^2}\,dx = \lim_{A\to\infty}\frac{-1}{x}\Big|_{11}^{A} = \lim_{A\to\infty}\left(\cancel{\frac{-1}{A}}^{\,0} - \frac{-1}{11}\right) = \frac{1}{11}$ and

$\lim_{A\to\infty}\int_{10}^{A}\frac{1}{x^2}\,dx = \lim_{A\to\infty}\frac{-1}{x}\Big|_{10}^{A} = \lim_{A\to\infty}\left(\frac{-1}{A} - \frac{-1}{10}\right) = \frac{1}{10}$ so

$$0.091 \approx \frac{1}{11} < \frac{1}{11^2} + \frac{1}{12^2} + \cdots < \frac{1}{10} = 0.1$$

---

c.) $\int_{101}^{\infty} \frac{1}{x^2}\,dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2}\,dx \rightarrow$

$\lim\limits_{A\to\infty} \int_{101}^{A} \frac{1}{x^2}\,dx = \lim\limits_{A\to\infty} \frac{-1}{x}\Big|_{101}^{A} = \lim\limits_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{101}\right) = \frac{1}{101}$ and

$\lim\limits_{A\to\infty} \int_{100}^{A} \frac{1}{x^2}\,dx = \lim\limits_{A\to\infty} \frac{-1}{x}\Big|_{100}^{A} = \lim\limits_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{100}\right) = \frac{1}{100}$ so

$$0.0099 \approx \frac{1}{101} < R_{100} < \frac{1}{100} = 0.01$$

---

d.) $R_n < \int_{n}^{\infty} \frac{1}{x^2}\,dx \le 0.0001 \rightarrow$

$\lim\limits_{A\to\infty} \int_{n}^{A} \frac{1}{x^2}\,dx \le 0.0001 \rightarrow \lim\limits_{A\to\infty} \frac{-1}{x}\Big|_{n}^{A} \le 0.0001$

$\rightarrow \lim\limits_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{n}\right) \le 0.0001 \rightarrow \frac{1}{n} \le 0.0001 \rightarrow$

$n \ge 10{,}000$.

3.) a.) $S_{10} = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^9 = \dfrac{1-\left(\frac{2}{3}\right)^{10}}{1-\left(\frac{2}{3}\right)} \approx 2.9480$

b.) $\int_{11}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx < R_{10} < \int_{10}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx \rightarrow$

$\lim\limits_{A\to\infty} \int_{11}^{A} \left(\frac{2}{3}\right)^{x-1} dx = \lim\limits_{A\to\infty} \dfrac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\Bigg|_{11}^{A}$

$= \lim\limits_{A\to\infty} \left(\cancelto{0}{\dfrac{\left(\frac{2}{3}\right)^{A-1}}{\ln(2/3)}} - \dfrac{\left(\frac{2}{3}\right)^{10}}{\ln(2/3)} \right) = \dfrac{-\left(\frac{2}{3}\right)^{10}}{\ln(2/3)} \approx 0.0427$ and

$\lim\limits_{A\to\infty} \int_{10}^{A} \left(\frac{2}{3}\right)^{x-1} dx = \lim\limits_{A\to\infty} \dfrac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\Bigg|_{10}^{A}$

$$= \lim_{A\to\infty}\left(\frac{\cancel{\left(\tfrac{2}{3}\right)^{A-1}}^{\,0}}{\ln(2/3)} - \frac{\left(\tfrac{2}{3}\right)^{9}}{\ln(2/3)}\right) = \frac{-\left(\tfrac{2}{3}\right)^{9}}{\ln(2/3)} \approx 0.0642 \text{ so}$$

$$0.0427 < R_{10} < 0.0642 .$$

---

c.) $R_{10} = \left(\frac{2}{3}\right)^{10} + \left(\frac{2}{3}\right)^{11} + \left(\frac{2}{3}\right)^{12} + \cdots = \left(\frac{2}{3}\right)^{10}\cdot\left[1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^{2} + \cdots\right]$

$= \left(\frac{2}{3}\right)^{10}\cdot\frac{1}{1-\left(\frac{2}{3}\right)} \approx 0.0520$

---

d.) $R_n < \int_n^\infty \left(\frac{2}{3}\right)^{x-1} dx \le 0.0001 \rightarrow$

$\lim_{A\to\infty} \int_n^A \left(\frac{2}{3}\right)^{x-1} dx \le 0.0001 \rightarrow$

$\lim_{A\to\infty} \left.\frac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\right|_n^A = \lim_{A\to\infty}\left(\frac{\cancel{\left(\tfrac{2}{3}\right)^{A-1}}^{\,0}}{\ln(2/3)} - \frac{\left(\tfrac{2}{3}\right)^{n-1}}{\ln(2/3)}\right) \le 0.0001 \rightarrow$

$\frac{-\left(\frac{2}{3}\right)^{n-1}}{\ln(2/3)} \le 0.0001 \rightarrow \left(\frac{2}{3}\right)^{n-1} \le -(0.0001)\ln(2/3) \rightarrow$

$(n-1)\ln\left(\frac{2}{3}\right) \le \ln[-(0.0001)\ln(2/3)] \rightarrow$

$n-1 \ge \frac{\ln[-(0.0001)\ln(2/3)]}{\ln(2/3)} \rightarrow$

$n \ge \frac{\ln[-(0.0001)\ln(2/3)]}{\ln(2/3)} + 1 \approx 25.9$ so

choose $n \ge 26$.

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e.) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \cdots$

$= \frac{1}{1-\left(\frac{2}{3}\right)} = 3 .$