Section 10.5

Using Ratio Test or Root Test:

17) $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}}{\frac{n^{\sqrt{2}}}{2^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} \cdot \left(1+\frac{1}{n}\right)^{\sqrt{2}}$

$= \frac{1}{2} \cdot (1)^{\sqrt{2}} = \frac{1}{2} < 1$, so series converges by ratio test

20) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} (n+1) \cdot \frac{1}{10} = \infty > 1$,

so series diverges by ratio test

21) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \cdot \frac{1}{10}$

$= (1)^{10} \cdot \frac{1}{10} = \frac{1}{10} < 1$, so series converges by ratio test.

28) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$; $\lim_{n\to\infty} \left[\frac{(\ln n)^n}{n^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1,$$ so series converges by root test

29) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$; $\displaystyle\lim_{n\to\infty} \left[\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right]^{1/n}$

$$= \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1,$$ so series converges by root test

34) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^n}$; $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{e^{n+1}}}{\frac{n^3}{e^n}}$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{e^n}{e^{n+1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{e}$$

$$= (1)^3 \cdot \frac{1}{e} = \frac{1}{e} < 1,$$ so series converges by ratio test

35) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+4)!}{3!(n+1)!\,3^{n+1}}}{\frac{(n+3)!}{3!\,n!\,3^n}}$$

$$= \lim_{n\to\infty} \frac{(n+4)!}{(n+3)!} \cdot \frac{\cancel{3!}}{\cancel{3!}} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} (n+4) \cdot \frac{1}{(n+1)} \cdot \frac{1}{3}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{4}{n}}{1 + \frac{1}{n}} \cdot \frac{1}{3} = \frac{1+0}{1+0} \cdot \frac{1}{3} = \frac{1}{3}$$

37) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$

$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{n+1}{4n^2+10n+6}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}} = \frac{0+0}{4+0+0} = 0 < 1$, so series converges by ratio test

38) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty} \frac{1^n}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$,

so series converges by ratio test

46) $a_1 = 1$, $a_{n+1} = \frac{1+\arctan n}{n} \cdot a_n$;

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1+\arctan n}{n} \cdot \cancel{a_n}}{\cancel{a_n}}$

$= \frac{1+\frac{\pi}{2}}{\infty} = 0 < 1$, so series converges by ratio test

47) $a_1 = \frac{1}{3}$, $a_{n+1} = \frac{3n-1}{2n+5} \cdot a_n$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3n-1}{2n+5} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{1}{n}}{2 + \frac{5}{n}} = \frac{3-0}{2+0} = \frac{3}{2} > 1,$$ so

series diverges by ratio test.

55) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1} \cdot (n+1)! \cdot (n+1)!}{(2(n+1))!}}{\frac{2^n \cdot n! \cdot n!}{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} 2 \cdot (n+1) \cdot (n+1) \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} 2 \frac{(n^2+2n+1)}{4n^2+6n+2} = \lim_{n\to\infty} \frac{\cancel{2}}{\cancel{2}} \cdot \frac{n^2+2n+1}{2n^2+3n+1}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{3}{n} + \frac{1}{n^2}} = \frac{1+0+0}{2+0+0} = \frac{1}{2} < 1,$$

so series converges by ratio test

58) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$; apply root test :

(*) $$\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n} = \boxed{?}$$;

this is a sneaky twist: consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;

apply ratio test: (SEE Problem 22)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \cdots = \frac{1}{e} < 1$$

so series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges;

thus, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (by contrapositive to nth-term test);

(*) so $\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n} = 0 < 1$,

so series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$ converges by root test

43.) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2(n+1)-1)}{4^{n+1}2^{n+1}(n+1)!}}{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} (2n+1)\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{8}\cdot\frac{2n+1}{n+1} = \lim_{n\to\infty} \frac{1}{8}\cdot\frac{2+\frac{1}{n}}{1+\frac{1}{n}}$$

$= \frac{1}{8}\cdot\frac{2+0}{1+0} = \frac{1}{4} < 1$, so series converges by ratio test

<u>Using Any Test</u> :

22) $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^n$; $\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-2)}{n}\right)^n$
$= e^{-2} = \frac{1}{e^2} \neq 0$, so series diverges by nth-term test

23) $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{1.25^n}$; $0 \le \frac{2+(-1)^n}{1.25^n} \le \frac{2+1}{1.25^n} = 3\left(\frac{4}{5}\right)^n$;
$\sum_{n=1}^{\infty} 3\left(\frac{4}{5}\right)^n$ converges by geometric series test ($r=\frac{4}{5}$, $-1<r<1$),
so $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{1.25^n}$ converges by comparison test

24) $\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$; $r = -\frac{2}{3}$,
$-1<r<1$, so series converges by geometric series test

26) $\sum_{n=1}^{\infty}\left(1-\frac{1}{3n}\right)^n$; $\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-1/3)}{n}\right)^n$

$= e^{-1/3} = \frac{1}{e^{1/3}} \neq 0$ so series diverges by nth-term test

27) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$; $\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by limit comparison test

29) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n^2}$

diverges by subtle facts about series, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \leq 1$) and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$).

31) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; $\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n$

$= \infty$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \leq 1$), so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by limit comparison test

39) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$; $\lim_{n\to\infty} \left[\frac{n}{(\ln n)^n}\right]^{1/n}$

$= \lim_{n\to\infty} \frac{n^{1/n}}{\ln n}$ (and $\lim_{n\to\infty} n^{1/n} = $ "∞^0"

(indeterminate) $= \lim_{n\to\infty} e^{\ln n^{1/n}}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1.$)

$= \frac{1}{\infty} = 0 < 1$, so series converges by root test