Section 10.6

1) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$; Let $a_n = \frac{1}{n^2}$, then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so

series converges by alternating series test

9) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$; Let $a_n = \left(\frac{n}{10}\right)^n$, then $\lim_{n\to\infty} \left(\frac{n}{10}\right)^n = \infty$ so $\lim_{n\to\infty} (-1)^{n+1}\left(\frac{n}{10}\right)^n \neq 0$,

and series diverges by the nth-term test

10) $\sum_{n=2}^{\infty}(-1)^{n+1}\cdot\frac{1}{\ln n}$; Let $a_n = \frac{1}{\ln n}$, then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty}\frac{1}{\ln n} = 0$, so series converges by alternating series test

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\ln n}{n}$; Let $a_n = \frac{\ln n}{n}$, then a_n is $+$ (for $n \geq 2$) and $\lim_{n\to\infty}\frac{\ln n}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$; $f(x) = \frac{\ln x}{x} \xrightarrow{D} f'(x) = \frac{x\cdot\frac{1}{x} - \ln x}{x^2}$

$= \frac{1-\ln x}{x^2}$

NO $\quad + \quad 0 \quad - \qquad f'$

$x=0 \quad x=e\approx 2.7$

so a_n is $+$ for $n \geq 3$, $\downarrow$ for $n \geq 3$, and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=3}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converges by alternating series test, so $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\ln n}{n}$ converges

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\sqrt{n}+1}{n+1}$; Let $a_n = \frac{\sqrt{n}+1}{n+1}$, then a_n is $+$ and $\lim_{n\to\infty}\frac{\sqrt{n}+1}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}}}{1} = 0$;

$f(x) = \frac{\sqrt{x}+1}{x+1} \xrightarrow{D} f'(x) = \frac{(x+1)\cdot\frac{1}{2\sqrt{x}} - (\sqrt{x}+1)(1)}{(x+1)^2}$

$$= \frac{\frac{x+1}{2\sqrt{x}} - \frac{2\sqrt{x}\,(\sqrt{x}+1)}{2\sqrt{x}}}{\frac{(x+1)^2}{1}} = \frac{(x+1)-(2x+2\sqrt{x})}{2\sqrt{x}\cdot(x+1)^2}$$

$$= \frac{1-x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2}$$

$-$ at $x=1$, $-$ after; f'

so a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$ so series converges by alternating series test

14) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3\frac{\sqrt{n+1}}{\sqrt{n}+1}$; Let $a_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$, then $\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\frac{3}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}}$

$= \lim_{n\to\infty} 3\frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} 3\sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} 3\sqrt{\frac{1}{1+\frac{1}{n}}}$

$= 3\sqrt{\frac{1}{1+0}} = 3(1) = 3$; thus,

$\lim_{n\to\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1} \neq 0$ so series diverges by nth-term test

16) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(0.1)^n}{n}$; consider series $\sum_{n=1}^{\infty} \frac{(0.1)^n}{n}$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(0.1)^{n+1}}{n+1}}{\frac{(0.1)^n}{n}}$

$$= \lim_{n\to\infty} \frac{(0.1)^{n+1}}{(0.1)^n} \cdot \frac{n}{n+1} = \lim_{n\to\infty} (0.1) \cdot \frac{1}{1+\frac{1}{n}}$$

$= (0.1) \cdot \frac{1}{1+0} = \frac{1}{10} < 1$, so series $\sum_{n=1}^{\infty} \frac{(0.1)^n}{n}$ converges by ratio test, and series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(0.1)^n}{n}$ converges absolutely

17) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is +, ↓, and $\lim_{n\to\infty} a_n = 0$; but the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test ($p = \frac{1}{2} \leq 1$), so series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ converges conditionally

19) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$; consider series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$, then $\lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3+1}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = \frac{1}{1+0} = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p=2>1$) then $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by limit comparison test ; so $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$ converges absolutely

20) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n!}{2^n}$; Let $a_n = \frac{n!}{2^n}$, then compare a_n and a_{n+1} :

$$\frac{n!}{2^n} \sim \frac{(n+1)!}{2^{n+1}} \text{ iff } \frac{2^{n+1}}{2^n} \sim \frac{(n+1)!}{n!}$$

iff $2 \sim n+1$; but $2 \le n+1$ for $n = 1, 2, 3, \ldots$, so $a_n \le a_{n+1}$; since $a_1 = \frac{1}{2}$ and $a_n \le a_{n+1}$, then $\lim_{n\to\infty} a_n \ne 0$, so $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n!}{2^n} \ne 0$ and series diverges

22) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\sin n}{n^2}$; consider series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$; $0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, so $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges by comparison test; thus $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\sin n}{n^2}$ converges absolutely

25) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1+n}{n^2}$ converges by the alternating series test since $a_n = \frac{1+n}{n^2} = \frac{1}{n^2} + \frac{1}{n}$ is $+$, $\downarrow$, and

$\lim_{n\to\infty} a_n = 0$; consider series $\sum_{n=1}^{\infty} \frac{1+n}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges
(by subtle facts about series)

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent p-series ($p = 2 > 1$) and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent p-series ($p = 1 \le 1$).

27) $\sum_{n=1}^{\infty} (-1)^n \cdot n^2 \cdot \left(\frac{2}{3}\right)^n$; consider series $\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{2}{3}\right)^n$; then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot \left(\frac{2}{3}\right)^{n+1}}{n^2 \cdot \left(\frac{2}{3}\right)^n}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \left(\frac{2}{3}\right) = (1)^2\left(\frac{2}{3}\right) = \frac{2}{3} < 1$,

so $\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{2}{3}\right)^n$ converges by ratio test; thus, $\sum_{n=1}^{\infty} (-1)^n \cdot n^2 \cdot \left(\frac{2}{3}\right)^n$ converges absolutely

28) $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n \ln n}$ converges by alternating series test since $a_n = \frac{1}{n \ln n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$;

consider series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; let $f(x) = \frac{1}{x \ln x}$, then f is $+$, $\downarrow$, and continuous for $x \ge 2$; thus,

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{A\to\infty} \int_2^A \frac{1}{x\ln x}\,dx$$

$$= \lim_{A\to\infty} \ln|\ln x|\Big|_2^A = \lim_{A\to\infty} \left(\ln|\overset{\infty}{\ln A}| - \ln|\ln 2|\right)$$

$= \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by integral test, and $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n\ln n}$ converges conditionally

31) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{n}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1} = 1$

so $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n+1} \neq 0$ and series diverges by the nth-term test

33) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{100^n}{n!}$; consider series $\sum_{n=1}^{\infty} \frac{100^n}{n!}$; then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}}$

$= \lim_{n\to\infty} \frac{100^{n+1}}{100^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} 100 \cdot \frac{1}{n+1} = 0 < 1$,

so $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ converges by ratio test

and $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ is absolutely convergent

36) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \frac{\cos\pi}{1} + \frac{\cos 2\pi}{2} + \frac{\cos 3\pi}{3} + \cdots$

$= \frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$;

Let $a_n = \frac{1}{n}$, then a_n is +, ↓, and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges by alternating series test; but series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p=1 \leq 1$), so $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges conditionally.

40) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$; consider series $\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2 \cdot 3^{n+1}}{(2(n+1)+1)!}}{\frac{(n!)^2 \cdot 3^n}{(2n+1)!}}$

$$= \lim_{n\to\infty} \frac{(n+1)!(n+1)! \cdot 3^{n+1} \cdot (2n+1)!}{n! \cdot n! \cdot 3^n \cdot (2n+3)!}$$

$$= \lim_{n\to\infty} (n+1)(n+1) \cdot 3 \cdot \frac{1}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{3n^2+6n+3}{4n^2+10n+6} = \lim_{n\to\infty} \frac{3+\frac{6}{n}+\frac{3}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}}$$

$= \frac{3+0+0}{4+0+0} = \frac{3}{4} < 1$, so $\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$ converges by ratio test and $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$ converges absolutely

41) $\sum_{n=1}^{\infty} (-1)^n \cdot (\sqrt{n+1} - \sqrt{n})$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot (\sqrt{n+1} - \sqrt{n}) \cdot \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which converges by the alternating series test since $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$; consider series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$; then $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$

$= \frac{1}{1+1} = \frac{1}{2}$; since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$), $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by limit comparison test, and $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1}-\sqrt{n})$ is conditionally convergent

50) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{10^n} = \underbrace{\frac{1}{10} + \frac{-1}{10^2} + \frac{1}{10^3} + \frac{-1}{10^4}}_{S_4 = 0.0909} + \frac{1}{10^5} + \frac{-1}{10^6} + \cdots$

$\uparrow$ error $R_4 < 0.00001$

so $S_4 = 0.0909$ estimates (under-estimate) the exact value of $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{10^n}$

with error at most $R_4 < 0.00001$

58) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!} = \underbrace{1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}}_{S_n} + (-1)^{n+1} \cdot \frac{1}{(n+1)!} + \cdots$

error $R_n < \frac{1}{(n+1)!}$

require that $\frac{1}{(n+1)!} < 5 \times 10^{-6} = 0.000005$;

by calculator: $\frac{1}{8!} \approx 2.5 \times 10^{-5}$,

$\frac{1}{9!} \approx 2.7 \times 10^{-6} < 5 \times 10^{-6}$;

so choose $n = 8$; then

$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!} \approx S_8 = \cancel{1} - \cancel{\frac{1}{1!}} + \frac{2}{2!} - \frac{3}{3!} + \frac{4}{4!} - \frac{5}{5!} + \frac{6}{6!} - \frac{7}{7!} + \frac{8}{8!}$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!}$

$= 0.632142857143$ and absolute error is at most 0.000005.

62) $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$ (I)

$S_1 = 1$,

$\rightarrow S_2 = 1 - \frac{1}{2} = \frac{1}{2}$,

$S_3 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = 1$,

$\rightarrow S_4 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = \frac{2}{3}$,

$S_5 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} = 1$,

$\rightarrow S_6 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = \frac{3}{4}$, ...

$S_{2n} = \frac{n}{n+1}$; $S_{2n+1} = 1$;

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots \qquad (\mathrm{II})$$

$S_1 = \frac{1}{2}$,

$S_2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$,

$S_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{3}{4}$,

$S_4 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20} = 4/5$, ...

$S_n = \frac{n}{n+1}$;

for series (I), since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

and $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} 1 = 1$, so sequence of partial sums converges to 1, so the series (I) has sum 1 ;

for series (II), the sequence of partial sums S_n satisfies $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$, so series (II) has sum 1.

66) Both $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{\sqrt{n}} = 1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$

and $\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{1}{\sqrt{n}} = -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}-\cdots$

converge by the alternating series test, but

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{\sqrt{n}}\cdot(-1)^{n}\cdot\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}(-1)\cdot\frac{1}{n} = -\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right.$$

diverges by the p-series test.