Section 10.7

1) a.) $\sum_{n=0}^{\infty} x^n$; $\lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} |x| = |x| < 1$

$\to$ $\underline{-1 < x < 1}$; $\underline{\text{check } x=1}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x=-1}$: $\sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since $\lim_{n\to\infty} (-1)^n \neq 0$; so interval of convergence is $\boxed{-1 < x < 1}$.

4) a.) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$; $\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{n+1} \cdot \frac{n}{|3x-2|^n}$

$= \lim_{n\to\infty} |3x-2| \cdot \frac{n}{n+1} = |3x-2| \cdot (1)$

$= |3x-2| < 1 \longrightarrow -1 < 3x-2 < 1 \longrightarrow$

$1 < 3x < 3 \to \underline{1/3 < x < 1}$; $\underline{\text{check } x=1}$:

$\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$; $\underline{\text{check } x = 1/3}$:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test since

$a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$; so interval of convergence is $\boxed{\frac{1}{3} \leq x < 1}$.

6) a.) $\sum_{n=0}^{\infty} (2x)^n$; $\lim_{n\to\infty} \frac{|2x|^{n+1}}{|2x|^n}$

$= \lim_{n\to\infty} |2x| = |2x| < 1 \to -1 < 2x < 1 \to$

$\underline{\frac{-1}{2} < x < \frac{1}{2}}$; $\underline{\text{check } x = 1/2}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x = -1/2}$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by nth-term test since $\lim_{n\to\infty} (-1)^n \neq 0$; so interval of convergence is $\boxed{\frac{-1}{2} < x < \frac{1}{2}}$.

7) a.) $\sum_{n=0}^{\infty} \frac{n x^n}{n+2}$; $\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+3} \cdot \frac{n+2}{n|x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+3} \cdot \frac{n+2}{n} = |x| \cdot (1) \cdot (1) = |x| < 1$

$\rightarrow \underline{-1 < x < 1}$; $\underline{\text{check } x = 1}$: $\sum_{n=0}^{\infty} \frac{n \cdot 1^n}{n+2}$

$= \sum_{n=0}^{\infty} \frac{n}{n+2}$ diverges by nth-term test since $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$;

$\underline{\text{check } x = -1}$: $\sum_{n=0}^{\infty} \frac{n(-1)^n}{n+2}$ diverges by the nth-term test since $\lim_{n\to\infty} \frac{n}{n+2} = 1$ and so $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n+2} \neq 0$;

so interval of convergence is

$\boxed{-1 < x < 1}$.

10) a.) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$; $\lim_{n\to\infty} \frac{|x-1|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x-1|^n}$

$= \lim_{n\to\infty} |x-1| \cdot \sqrt{\frac{n}{n+1}} = |x-1| \cdot \sqrt{1} = |x-1| < 1$

$\to -1 < x-1 < 1 \to \underline{0 < x < 2}$; check $x=0$:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; check $x=2$:

$\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test $(p = \frac{1}{2} \le 1)$; so interval of convergence is

$\boxed{0 \le x < 2}$

12) a.) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$; $\lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n |x|^n}$

$= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} |x| = 3 \cdot (0) \cdot |x| = 0 < 1$

for all x-values, so interval of convergence is $\boxed{-\infty < x < \infty}$.

17) a.) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$; $\lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x+3|^n}$

$= \lim_{n\to\infty} \frac{1}{5} \cdot \left(1 + \frac{1}{n}\right)|x+3| = \frac{1}{5} \cdot (1) \cdot |x+3|$

$= \frac{1}{5}|x+3| < 1 \to |x+3| < 5 \to -5 < x+3 < 5$

$\to \underline{-8 < x < 2}$; check $x=2$:

$\sum_{n=0}^{\infty} n \cdot \frac{\cancel{5^n}}{\cancel{5^n}} = \sum_{n=0}^{\infty} n$ diverges by nth-term test since $\lim_{n\to\infty} n = \infty \neq 0$;

<u>check $x=-8$</u>: $\sum_{n=0}^{\infty} n \cdot \frac{(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n$

diverges by the nth-term test since $\lim_{n\to\infty} (-1)^n \cdot n \neq 0$; so interval of convergence is $\boxed{-8 < x < 2}$.

24) a.) $\sum_{n=1}^{\infty} (\ln n) x^n$; $\lim_{n\to\infty} \frac{\ln(n+1) \cdot |x|^{n+1}}{\ln n \cdot |x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{\ln n+1}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} |x| \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}}$

$= \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| \cdot (1) = |x| < 1 \longrightarrow$

<u>$-1 < x < 1$</u>; <u>check $x=1$</u>: $\sum_{n=1}^{\infty} (\ln n) \cdot 1^n$

$= \sum_{n=1}^{\infty} (\ln n)$ diverges by nth-term test since $\lim_{n\to\infty} \ln n = \infty \neq 0$;

<u>check $x=-1$</u>: $\sum_{n=1}^{\infty} (\ln n) \cdot (-1)^n$ diverges by nth-term test since $\lim_{n\to\infty} \ln n = \infty$, so $\lim_{n\to\infty} (-1)^n \cdot \ln n \neq 0$;

so interval of convergence is $\boxed{-1 < x < 1}$.

25) a.) $\sum_{n=1}^{\infty} n^n x^n$; $\lim_{n\to\infty} \frac{(n+1)^{n+1}\cdot|x|^{n+1}}{n^n\cdot|x|^n}$

$= \lim_{n\to\infty} (n+1)\cdot\frac{(n+1)^n}{n^n}\cdot|x|$

$= \lim_{n\to\infty} (n+1)\cdot\left(1+\frac{1}{n}\right)^n\cdot|x| = \begin{cases}\infty(1)\cdot|x| = \infty, \text{ if } x\neq 0\\ 0<1 \text{ if } x=0\end{cases}$;

so interval of convergence is $\boxed{x=0}$.

27) a.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$;

$\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)\cdot 2^{n+1}}\cdot\frac{n\cdot 2^n}{|x+2|^n} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{2}\cdot|x+2|$

$= (1)\cdot\frac{1}{2}\cdot|x+2| = \frac{1}{2}|x+2| < 1 \longrightarrow$

$|x+2| < 2 \longrightarrow -2 < x+2 < 2 \longrightarrow$

$\underline{-4 < x < 0}$; $\underline{\text{check } x=0}$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\cancel{2^n}}{n\cdot\cancel{2^n}}$

$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by the alternating series test since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty}\frac{1}{n} = 0$; $\underline{\text{check } x=-4}$:

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot(-1)^n\cdot\frac{1}{n}$

$= \sum_{n=1}^{\infty}(-1)\cdot(-1)^n\cdot(-1)^n\cdot\frac{1}{n} = \sum_{n=1}^{\infty}(-1)(-1)^{2n}\cdot\frac{1}{n}$

$= \sum_{n=1}^{\infty}(-1)\,(\underbrace{(-1)^2}_{1})^n\cdot\frac{1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$; so interval of convergence is $\boxed{-4 < x \leq 0}$.

29) a.) $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$; $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{|x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^2$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} |x| \cdot \frac{1}{1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n+1}}\right)^2 = \lim_{n\to\infty} |x| \cdot \left(\frac{n+1}{n}\right)^2$

$= |x| \cdot (1)^2 = |x| < 1 \longrightarrow \underline{-1 < x < 1}$;

<u>check $x = 1$</u> : $\sum_{n=2}^{\infty} \frac{1^n}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$;

$f(x) = \frac{1}{x(\ln x)^2}$ is $+$, $\downarrow$, and continuous for $x \geq 2$ and

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{A\to\infty} \frac{-1}{\ln x}\Big|_2^A$

$= \lim_{x\to\infty} \left(\frac{-1}{\ln A} - \frac{-1}{\ln 2}\right) = \frac{1}{\ln 2}$ (converges)

so series converges by integral test,

<u>check $x = -1$</u> : $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges

by the alternating series test since $a_n = \frac{1}{n(\ln n)^2}$ is $+$, $\downarrow$, and

$\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$; so interval of convergence is $\boxed{-1 \leq x \leq 1}$.

44) $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$; $\lim_{n\to\infty} \frac{|x+1|^{2(n+1)}}{9^{n+1}} \cdot \frac{9^n}{|x+1|^{2n}}$

$= \lim_{n\to\infty} \frac{1}{9}|x+1|^2 = \frac{1}{9}|x+1|^2 < 1 \longrightarrow$

$|x+1|^2 < 9 \longrightarrow |x+1| < 3 \rightarrow$

$-3 < x+1 < 3 \longrightarrow \underline{-4 < x < 2};$

<u>check $x = 2$</u>: $\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n}$

$= \sum_{n=0}^{\infty} 1$ diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; <u>check $x = -4$</u>:

$\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n} = \sum_{n=0}^{\infty} 1$ diverges

by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; so interval of convergence is

$\boxed{-4 < x < 2}$;

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\frac{(x+1)^2}{9}\right)^n$$

$$= 1 + \left(\frac{(x+1)^2}{9}\right) + \left(\frac{(x+1)^2}{9}\right)^2 + \left(\frac{(x+1)^2}{9}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{(x+1)^2}{9}} \cdot \frac{9}{9} = \frac{9}{9-(x^2+2x+1)} = \frac{9}{8-x^2-2x}$$

46) $\sum_{n=0}^{\infty} (\ln x)^n$; $\lim_{n\to\infty} \frac{|\ln x|^{n+1}}{|\ln x|^n}$

$= \lim_{n\to\infty} |\ln x| = |\ln x| < 1 \longrightarrow$

$-1 < \ln x < 1 \longrightarrow e^{-1} < x < e^1 \longrightarrow$

$\frac{1}{e} < x < e$; check $x=e$: $\sum_{n=0}^{\infty} (\ln e)^n$

$= \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges by

nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$;

check $x = \frac{1}{e}$: $\sum_{n=0}^{\infty} (\ln e^{-1})^n = \sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since

$\lim_{n\to\infty} (-1)^n \neq 0$; so interval of

convergence is $\boxed{\frac{1}{e} < x < e}$;

$\sum_{n=0}^{\infty} (\ln x)^n = 1 + (\ln x) + (\ln x)^2 + (\ln x)^3 + \cdots$

$= \frac{1}{1 - \ln x}$.

56) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$; begin with

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \xrightarrow{D}$

$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \longrightarrow$

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots \xrightarrow{D}$

$\frac{(1-x)^2 \cdot (1) - x \cdot 2(1-x)(-1)}{(1-x)^4} = 1 + 4x + 9x^2 + 16x^3 + \cdots$

$\longrightarrow$

$$\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots \longrightarrow$$

Let $x = \frac{1}{2}$

$$\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots \longrightarrow$$

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + \cdots$$

$$= \frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{n^2}{2^n} \longrightarrow$$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \frac{3/4}{1/8} = \frac{3}{4} \cdot \frac{8}{1} = 6 .$$