Section 10.9

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow$

$e^{-5x} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots$

$= 1 - 5x + \frac{5^2}{2!}x^2 - \frac{5^3}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{n!} x^n$

12) $x^2 \sin x = x^2 \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$

$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2n+1}}{(2n-1)!}$

16) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \longrightarrow$

$x^2 \cos(x^2) = x^2 \cdot \left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right]$

$= x^2 \cdot \left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right]$

$= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+2}}{(2n)!}$

17) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{2}\left(1 + \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)\right.$

$$= \frac{1}{2}\left(2 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots\right)$$
$$= 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!}x^6 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty}(-1)^n \cdot \frac{2^{2n-1}}{(2n)!}x^{2n} \qquad \text{OR}$$

$$\cos^2 x = \cos x \cdot \cos x$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$- \frac{x^2}{2!} + \frac{x^4}{4} - \frac{x^6}{48} + \cdots$$
$$+ \frac{x^4}{4!} - \frac{x^6}{48} + \cdots$$
$$- \frac{x^6}{6!} + \cdots$$
$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

19) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \longrightarrow$

$$\frac{x^2}{1-2x} = x^2 \cdot \frac{1}{1-(2x)} = x^2\left[1 + (2x) + (2x)^2 + (2x)^3 + \cdots\right]$$
$$= x^2 + 2x^3 + 2^2 \cdot x^4 + 2^3 \cdot x^5 + \cdots = \sum_{n=1}^{\infty} 2^{n-1} \cdot x^{n+1}$$

20) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \longrightarrow$

$$x \cdot \ln(1+(2x)) = x \cdot \left[(2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots\right]$$

$$= x \cdot \left[2x - \frac{2^2}{2}x^2 + \frac{2^3}{3}x^3 - \frac{2^4}{4}x^4 + \cdots\right]$$
$$= 2x^2 - \frac{2^2}{2}x^3 + \frac{2^3}{3}x^4 - \frac{2^4}{4}x^5 + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n} x^{n+1}$$

21) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \xrightarrow{D}$

$D(1-x)^{-1} = D(1 + x + x^2 + x^3 + x^4 + \cdots) \rightarrow$

$-1(1-x)^{-2} \cdot (-1) = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \rightarrow$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$

OR

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x}$$
$$= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$
$$\begin{aligned} = 1 + x &+ x^2 + x^3 + \cdots \\ + x &+ x^2 + x^3 + \cdots \\ &+ x^2 + x^3 + \cdots \\ &\qquad + x^3 + \cdots \end{aligned}$$
$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

22) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \xrightarrow{D}$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \xrightarrow{D}$

$$\frac{2}{(1-x)^3} = 2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+2)(n+1) x^n$$

$$R_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}, \quad \text{where } c_n$$

is between x and a

37) $$\sin x = \underbrace{x - \frac{x^3}{3!}}_{} + \underbrace{\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}_{R_4(x;0)}$$

$!! \to P_4(x;0) = P_3(x;0)$

$$|R_4(x;0)| = \left| \frac{f^{(5)}(c_4) \cdot (x-0)^5}{5!} \right|, \quad c_4 \text{ is between } x \text{ and } 0$$

$$= \frac{|\cos(c_4)| \cdot |x|^5}{5!}$$

$$\leq \frac{1 \cdot |x|^5}{5!}$$

$$= \frac{|x|^5}{120}; \quad \text{require that}$$

$$\frac{|x|^5}{120} \leq 5 \cdot 10^{-4} = 0.0005 \longrightarrow$$

$$|x|^5 \leq 0.06 \longrightarrow$$

$$|x| \leq (0.06)^{1/5} \approx 0.5696 \longrightarrow$$

$$-0.5696 \leq x \leq 0.5696$$

40) $$f(x) = (1+x)^{1/2} \xrightarrow{D} f'(x) = \frac{1}{2}(1+x)^{-1/2} \xrightarrow{D}$$

$$f''(x) = \frac{-1}{2^2}(1+x)^{-3/2} \xrightarrow{D} f'''(x) = \frac{3}{2^3} \cdot (1+x)^{-\frac{5}{2}} \xrightarrow{D}$$

$$f^{(4)}(x) = -\frac{3 \cdot 5}{2^4} \cdot (1+x)^{-7/2} \xrightarrow{D}$$

$$f^{(5)}(x) = \frac{3 \cdot 5 \cdot 7}{2^5}(1+x)^{-9/2} \longrightarrow \cdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}\cdot(1\cdot3\cdot5\cdots(2n-3))}{2^n}(1+x)^{-\frac{(2n-1)}{2}}$$

for $n = 2, 3, 4, 5, \ldots$; then

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{\frac{1}{2}}{1} = \frac{1}{2},$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2^2\cdot2!}, \quad a_3 = \frac{f'''(0)}{3!} = \frac{3}{2^3\cdot3!},$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-3\cdot5}{2^4\cdot4!}, \ldots,$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}(1\cdot3\cdot5\cdots(2n-3))}{2^n\cdot n!}$$ for $n = 2, 3, 4, \ldots$;

$$(1+x)^{1/2} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$
$$= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}(1\cdot3\cdot5\cdots(2n-3))}{2^n\cdot n!}x^n$$

$$(1+x)^{1/2} = \underbrace{1 + \frac{1}{2}x}_{P_1(x;0)} \underbrace{- \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots}_{R_1(x;0)}$$

$$|R_1(x;0)| = \left|\frac{f''(c_1)\cdot(x-0)^2}{2!}\right|$$

$$\left(f(x) = (1+x)^{1/2} \xrightarrow{D} f'(x) = \frac{1}{2}(1+x)^{-1/2} \xrightarrow{D} f''(x) = \frac{-1}{4}(1+x)^{-3/2}\right)$$

$$= \left|\frac{\frac{-1}{4}(1+c_1)^{-3/2}\cdot x^2}{2!}\right| = \frac{1}{8}\cdot\frac{|x|^2}{|1+c_1|^{3/2}}$$

(where c_1 is between x and 0)

$$\leq \frac{1}{8} \cdot \frac{|0.01|^2}{|1+(-0.01)|^{3/2}} \quad (\text{since } -0.01 < x < 0.01)$$

$$= \frac{1}{8} \cdot \frac{(0.01)^2}{(0.99)^{3/2}}$$

$$\approx 0.0000127$$

41) $$e^x = \underbrace{1 + x + \frac{x^2}{2}}_{P_2(x;0)} + \underbrace{\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}_{R_2(x;0)}$$

$$|R_2(x;0)| = \left| \frac{f^{(3)}(c_2) \cdot (x-0)^3}{3!} \right|, \quad c_2 \text{ is between } x \text{ and } 0$$

$$= \frac{e^{c_2}}{6} \cdot |x|^3$$

$$\leq \frac{e^{0.1}}{6} (0.1)^3 \quad (\text{since } -0.1 < x < 0.1)$$

$$< \frac{3^{0.1}}{6} (0.1)^3 \approx 0.000186$$

48) $\frac{1}{1-x} = \underbrace{1 + x + x^2 + x^3}_{P_3(x;0)} + \underbrace{x^4 + x^5 + \cdots}_{R_3(x;0)}$

$f(x) = \frac{1}{1-x} = (1-x)^{-1} \xrightarrow{D}$

$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} \xrightarrow{D}$

$f''(x) = -2(1-x)^{-3} \cdot (-1) = 2(1-x)^{-3} \xrightarrow{D}$

$f'''(x) = -6(1-x)^{-4} \cdot (-1) = 6(1-x)^{-4} \xrightarrow{D}$

$f^{(4)}(x) = -24(1-x)^{-5} \cdot (-1) = \frac{24}{(1-x)^5}$; then

$|R_3(x;0)| = \left| \frac{f^{(4)}(c_3)}{4!} \cdot (x-0)^4 \right| = \frac{\cancel{24}}{\cancel{24}} \cdot \frac{|x|^4}{|1-c_3|^5}$

$\leq \frac{|0.1|^4}{|1-0.1|^5}$ (since $-0.1 < x < 0.1$ and c_3 is between x and 0)

≈ 0.000169