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CONVERGENCE OF WEYL-HEISENBERG FRAME SERIES

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The concept of frames for $L^2(\mathbb{R})$ is a generalization of the concept of orthonormal bases. The Weyl-Heisenberg frames are a kind of important frames. It is generated by translations and modulations of a single function and is extensively used in time frequency analysis. Here we study the convergence of the Weyl-Heisenberg frame series.

Key words: Uniform convergence, frame series, dual frame, Weyl-Heisenberg frame

1. INTRODUCTION

The concept of frames for $L^2(\mathbb{R})$ is first introduced by Duffin and Schaeffer in [5]. Weyl-Heisenberg frames are generated by translations and modulations of a single function. They form an important class of frames which are used extensively in time-frequency analysis.

It is well-known that $\{e^{i2\pi mt}\chi_{[0,1]}(t-n)\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, where $\chi_{[0,1]}$ is the characteristic function of $[0,1]$. Replacing $\chi_{[0,1]}$ by a general function W, m by $m\omega_0$, and n by nt_0, this leads to the concept of Weyl-Heisenberg frame with the modulation parameter ω_0, the translation parameter t_0, and the mother function W (simply, W-H frame).

The terminology, W-H frame, comes from the representation of the Weyl-Heisenberg group [1]. It is also called Gabor frame since it is obtained by discretization of the Gabor transform in communication [4, 6].

W-H frames are studied by many authors. Ron and Shen [8] gave a complete characterization of the W-H frames, Casazza and Christensen [2] gave an estimate of bounds of the W-H frames. Any $f \in L^2(\mathbb{R})$ can be expanded into W-H frame series. In this paper, we will study the convergence of the W-H frame series in detail. We will study the pointwise convergence of the Weyl-Heisenberg frame series and the uniform convergence of the compactly supported Weyl-Heisenberg frame series as well as their convergence rate.

2. Preliminary

Throughout this paper, $(\cdot\,,\,\cdot)$ expresses the inner product, $\|\cdot\|$ expresses the norm in $L^2(\mathbb{R})$, and

$$\{g_n\} = \{g_n\}_{n\in\mathbb{Z}}, \quad \{h_{m,n}\} = \{h_{m,n}\}_{m,n\in\mathbb{Z}},$$

$$\sum_n a_n = \sum_{n\in\mathbb{Z}} a_n, \quad \sum_{m,n} b_{m,n} = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} b_{m,n}.$$

We recall some known definitions and results.

Definition 2.1 — [1, 2, 6]. Let $\{\varphi_n\} \subset L^2(\mathbb{R})$. If there exist two positive constants $A,\ B$ such that

$$A \parallel f \parallel^2 \leq \sum_n |(f,\varphi_n)|^2 \leq B \parallel f \parallel^2 \qquad \forall f \in L^2(\mathbb{R}),$$

then $\{\varphi_n\}$ is called a frame of $L^2(\mathbb{R})$ with bounds A and B.

Definition 2.2 — [2, 6]. Let $W \in L^2(\mathbb{R})$ and $\omega_0,\ t_0 > 0$. Denote by $W_{m,n}$ the family

$$W_{m,n}(t) :\, = e^{im\omega_0 t} W(t - nt_0) \quad (m, n \in \mathbb{Z}). \tag{2.1}$$

If $\{W_{m,n}\}$ is a frame for $L^2(\mathbb{R})$, then $\{W_{m,n}\}$ is called a Weyl-Heisenberg frame, simply, W-H frame.

Definition 2.3 — [2, 6]. Let $\{W_{m,n}\}$ and $\{\widetilde{W}_{m,n}\}$ be two W-H frames. If, for any $f \in L^2(\mathbb{R})$,

$$f = \sum_{m,n} (f, W_{m,n}) \widetilde{W}_{m,n} \quad (L^2) \qquad \text{and} \qquad f = \sum_{m,n} (f, \widetilde{W}_{m,n}) W_{m,n} \quad (L^2),$$

then $\{W_{m,n}\ \widetilde{W}_{m,n}\}$ is called a pair of dual W-H frames.

The series $\sum\limits_{m,n} (f,\ W_{m,n}) \widetilde{W}_{m,n}(t)$ and $\sum\limits_{m,n} (f,\ \widetilde{W}_{m,n}) W_{m,n}(t)$ are both called the W-H frame series of f. By symmetry, we only study the convergence of the series $\sum\limits_{m,n} (f,\ W_{m,n}) \widetilde{W}_{m,n}(t)$ in this paper.

For a compactly supported W-H frame, its canonical dual W-H frame is constructed as follows.

Proposition 2.4 — [6] Suppose that W is continuous on $\mathbb{R}$ and $[a, b]$ is a finite interval. Let

$$W(t) \neq 0 \quad (t \in (a, b)),\ W(t) = 0\ (t \notin (a, b)) and\ 0 < \omega_0 \leq \frac{2\pi}{b-a},\ 0 < t_0 < b-a \tag{2.2}$$

and let

$$\widetilde{W}(t) = \frac{1}{G(t)} W(t), \quad \text{where} \quad G(t) = \frac{2\pi}{\omega_0} \sum_n |W(t - nt_0)|^2. \tag{2.3}$$

Then $\{W_{m,n},\ \widetilde{W}_{m,n}\}$ is a pair of dual W-H frames.

Proposition 2.5 — [3] Let $f \in L(\mathbb{R})$. If $\sum\limits_k f(x + 2k\pi)$ converges to a continuous function of bounded variation on $[0, 2\pi]$, then for each $x \in \mathbb{R}$, $\sum\limits_k f(x + 2k\pi) = \frac{1}{2\pi} \sum\limits_n \widehat{f}(n) e^{inx}$, where $\widehat{f}(\omega) = \int\limits_{\mathbb{R}} f(t) e^{-it\omega} \mathrm{d}t$ is the Fourier transform of f.

3. Pointwise Convergence of the W-H Frame Series

Let $\{W_{m,n},\ \widetilde{W}_{m,n}\}$ be a pair of dual W-H frames. By Definition 2.2, for any $f \in L^2(\mathbb{R})$, we have

$$f = \sum_{m,n} (f,\ W_{m,n}) \widetilde{W}_{m,n} \qquad (L^2). \tag{3.1}$$

***Theorem* 3.1** — *Let $\{W_{m,n},\ \widetilde{W}_{m,n}\}$ be a pair of dual W-H frames. Suppose that W is a continuous function of bounded variation on $\mathbb{R}$ and there exists a constant $C > 0$ such that for $t \in \mathbb{R}$,*

$$|W(t)|, \quad |\widetilde{W}(t)| \leq C \left(\frac{1}{(1 + |t|)^\alpha} \right) \quad \text{for some } \alpha > 1. \tag{3.2}$$

If $f \in L(\mathbb{R})$ is a continuous function of bounded variation on $\mathbb{R}$, then for each $t \in \mathbb{R}$,

$$f(t) = \sum_n \left(\sum_m (f,\ W_{m,n}) \widetilde{W}_{m,n}(t) \right).$$

PROOF : First, we prove that for each $n \in \mathbb{Z}$, the series $\sum\limits_m (f,\ W_{m,n}) \widetilde{W}_{m,n}(t)$ converges everywhere on $\mathbb{R}$.

By (2.1) and the definition of the Fourier transform, we have

$$(f,\ W_{m,n}) = \int\limits_{\mathbb{R}} f(t) \overline{W_{m,n(t)}} \mathrm{d}t = \int\limits_{\mathbb{R}} h_n(t) e^{-imt} \mathrm{d}t = \widehat{h}_n(m),$$

where

$$h_n(t) = \frac{1}{\omega_0} f\left(\frac{t}{\omega_0}\right) \overline{W\left(\frac{t}{\omega_0} - nt_0\right)}. \tag{3.3}$$

From this and (2.1), it follows that

$$(f,\ W_{m,n})\widetilde{W}_{m,n}(t)=\widehat{h}_n(m)e^{im\omega_0 t}\widetilde{W}(t-nt_0). \tag{3.4}$$

Now, we discuss the pointwise convergence of the series $\sum\limits_m \widehat{h}_n(m)e^{im\omega_0 t}$.

By the assumption and (3.3), we have $h_n \in L(\mathbb{R})$. Furthermore, $\int\limits_0^{2\pi}(\sum\limits_{k\in\mathbb{Z}}|h_n(t+2k\pi)|)\mathrm{d}t = \int\limits_{\mathbb{R}}|h_n(t)|\mathrm{d}t<\infty$. So, for each $n\in\mathbb{Z}$, the series $\sum\limits_k |h_n(t+2k\pi)|$ converges almost everywhere on $[0,2\pi]$. Take a point $t_0\in[0,2\pi]$ such that $\sum\limits_k |h_n(t_0+2k\pi)|$ converges. Arbitrarily take $t\in[0,2\pi]$. We have

$$\sum_{|k|\ge N}|h_n(t+2k\pi)|\le \sum_{|k|\ge N}|h_n(t_0+2k\pi)|+\sum_{|k|\ge N}|h_n(t+2k\pi)-h_n(t_0+2k\pi)| =: I_N(t_0)+J_N(t),$$

where $I_N(t_0)\to 0$. By the assumption and (3.3), we know that h_n is a continuous function of bounded variation on $\mathbb{R}$. For any $t\in[0,2\pi]$, $J_N(t)$ is not large than the total variation of h_n on $[2N\pi,\infty)$. So we obtain that $\lim\limits_{N\to\infty} J_N(t)=0$ uniformly on $[0,2\pi]$. Therefore, $\sum\limits_k h_n(t+2k\pi)$ uniformly converges on $[0,2\pi]$.

Denote $H_n(t)=\sum\limits_k h_n(t+2k\pi)$. We see that H_n is a $2\pi-$periodic continuous function and is of bounded variation on $[0,2\pi]$. By Proposition 2.5, we obtain that for each $t\in\mathbb{R}$,

$$H_n(\omega_0 t)=\frac{1}{2\pi}\sum_m \widehat{h}_n(m)e^{im\omega_0 t}.$$

Again, by (3.4), for each $n\in\mathbb{Z}$, we know that the series $\sum\limits_m (f,\ W_{m,n})\widetilde{W}_{m,n}(t)$ is convergent on $\mathbb{R}$ and

$$P_n(t):=\sum_m (f,W_{m,n})\widetilde{W}_{m,n}(t)=\widetilde{W}(t-nt_0)\sum_m \widehat{h}_n(m)e^{im\omega_0 t}=2\pi H_n(\omega_0 t)\widetilde{W}(t-nt_0)\quad (t\in\mathbb{R}). \tag{3.5}$$

Next, we prove that $\sum\limits_n P_n(t)$ converges on $\mathbb{R}$.

By (3.3), we have

$$H_n(\omega_0 t)=\sum_k h_n(\omega_0 t+2k\pi)=\frac{1}{\omega_0}\sum_k f\left(t+\frac{2k\pi}{\omega_0}\right)\overline{W(t-nt_0+\frac{2k\pi}{\omega_0})}.$$

From this and (3.5), we have

$$\sum_n |P_n(t)|=\sum_n 2\pi|H_n(\omega_0 t)\widetilde{W}(t-nt_0)|\le \frac{2\pi}{\omega_0}\sum_k \beta(k,\ t)\left|f(t+\frac{2k\pi}{\omega_0})\right|, \tag{3.6}$$

where $\beta(k,\ t) = \sum\limits_n |W(t - nt_0 + \frac{2k\pi}{\omega_0})\widetilde{W}(t - nt_0)|$.

By (3.2), we obtain that for some $\alpha > 1$,

$$\begin{aligned}\beta(k,\ t) &\leq C^2 \left(\sum_{|t-nt_0| \geq \frac{|k|\pi}{\omega_0}} + \sum_{|t-nt_0| < \frac{|k|\pi}{\omega_0}} \right) \frac{1}{(1+|t-nt_0+\frac{2k\pi}{\omega_0}|)^\alpha (1+|t-nt_0|)^\alpha} \\ &\leq \frac{C^2}{(1+\frac{|k|\pi}{\omega_0})^\alpha} \left(\sum_n \frac{1}{(1+|t-nt_0+\frac{2k\pi}{\omega_0}|)^\alpha} + \sum_n \frac{1}{(1+|t-nt_0|)^\alpha} \right) = O\left(\frac{1}{(1+|k|)^\alpha} \right).\end{aligned}$$

Again, since $f(t)$ is bounded on $\mathbb{R}$, by (3.6), $\sum\limits_n P_n(t)$ converges everywhere on $\mathbb{R}$. Denote its sum by $g(t)$, i.e.,

$$g(t) = \sum_n \left(\sum_m (f, W_{m,n}) \widetilde{W}_{m,n}(t) \right) \quad (t \in \mathbb{R}). \tag{3.7}$$

Finally, we will prove $g(t) = f(t)\ (t \in \mathbb{R})$.

Since $f \in L(\mathbb{R})$ is bounded on $\mathbb{R}$, we see that $f \in L^2(\mathbb{R})$. Again, by (3.1), we have

$$f = \sum_n \left(\sum_m (f, W_{m,n}) \widetilde{W}_{m,n} \right) \quad (L^2). \tag{3.8}$$

The Riesz theorem [9] shows that if a sequence of functions both converges in $L^2(\mathbb{R})$ and converges almost everywhere in $\mathbb{R}$, then their limiting functions are identical. Hence, by (3.7) and (3.8), $f(t) = g(t)\ (t \in \mathbb{R})$. Theorem 3.1 is proved. □

4. Uniform Convergence of the Compactly Supported W-H Frame Series

Theorem 4.1 — *Let $\{W_{m,n},\ \widetilde{W}_{m,n}\}$ be a pair of compactly supported dual W-H frames satisfying (2.2) and (2.3), and let W be continuous on $\mathbb{R}$. If $f \in L^2(\mathbb{R})$ is bounded continuous on $\mathbb{R}$, then for any $l \in \mathbb{Z}^+$, when $N > \frac{1}{t_0}(l + |a| + |b|)$* and *$M \in \mathbb{Z}^+$, the partial sums*

$$S_{M,N}(t) := \sum_{|n| \leq N} \sum_{|m| \leq M} (f,\ W_{m,n}) \widetilde{W}_{m,n}(t). \tag{4.1}$$

of the W-H frame series (3.1) satisfies

$$S_{M,N}(f;\ t) - f(t) = O\left(\frac{\| W \|_\infty}{m^* t_0 \omega_0} \right) \left(\| W \|_\infty\, \omega(f; \frac{1}{\omega_0 M}) + \| f \|_\infty\, \omega(W; \frac{1}{\omega_0 M}) \right) \log M$$

$$(|t| \leq l), \tag{4.2}$$

where $\omega(g;\ \delta) = \sup_{|t_1-t_2|\le\delta} |g(t_1) - g(t_2)|$ *is the modulus of continuity and*

$$m^* = \min_{|t-\frac{a+b}{2}|\le\frac{t_0}{2}} |W(t)|^2, \qquad \| g \|_\infty = \max_{t\in\mathbb{R}} |g(t)|.$$

In this section, the bounds in terms "O " are absolute constants.

PROOF : Since $W(t) = 0$ $(\omega \notin (a,b))$and $W(t)$ is a continuous function, we see that there are only finitely many nonzero terms in the series of (2.3) and $G(t)$ is a t_0-periodic continuous function. From this and (2.3), it follows that

$$\min_{t\in\mathbb{R}} G(t) = \min_{|t-\frac{a+b}{2}|\le\frac{t_0}{2}} G(t) \ge \frac{2\pi}{\omega_0} \min_{|t-\frac{a+b}{2}|\le\frac{t_0}{2}} |W(t)|^2 = \frac{2\pi}{\omega_0} m^* > 0. \tag{4.3}$$

By (2.1) and (2.3), we have $\widetilde{W}_{m,n}(t) = \frac{1}{G(t)} W_{m,n}(t)$. Again, by (4.1) and (2.1), we obtain that for any $f \in L^2(\mathbb{R})$,

$$G(t)S_{M,N}(f;\ t) = \sum_{|n|\le N} \sum_{|m|\le M} (f,\ W_{m,n})W_{m,n}(t) = \sum_{|n|\le N} J_{M,n}(t)W(t-nt_0), \tag{4.4}$$

where $J_{M,n}(t) = \int_{\mathbb{R}} f(s)\overline{W(s-nt_0)}D_M(\omega_0(t-s))ds$ and $D_M(u) = \sum_{|m|\le M} e^{imu}$.

By (2.2), we have

$$J_{M,n}(t) = \frac{1}{\omega_0} \int_{(a+nt_0)\omega_0}^{(a+nt_0)\omega_0+2\pi} F_n(s)D_M(\omega_0 t - s)\ ds \quad \text{and} \quad F_n(s) = f\left(\frac{s}{\omega_0}\right)\overline{W\left(\frac{s}{\omega_0} - nt_0\right)}. \tag{4.5}$$

Since $F_n(s) = 0$ $(s \notin ((a+nt_0)\omega_0,\ (a+nt_0)\omega_0 + 2\pi))$, we can extend $F_n(s)$ to a periodic continuous function $F_n^*(s)$ with period 2π, i.e.,

$$F_n^*(s+2k\pi) = F_n^*(s)\ (k\in\mathbb{Z}) \quad \text{and} \quad F_n^*(s) = F_n(s) \quad (\ (a+nt_0)\omega_0 \le s \le (a+nt_0)\omega_0+2\pi\ ). \tag{4.6}$$

Again noticing that $D_M(u)$ is an even periodic function with period 2π, we have

$$J_{M,n}(t) = \frac{1}{\omega_0} \int_0^{2\pi} F_n^*(s)D_M(s-\omega_0 t))\ ds. \tag{4.7}$$

On the other hand, since F_n^* is a continuous function with period 2π, by a known result [7, 10], we know that the partial sums of the Fourier series of F_n^*,

$$s_M(F_n^*,\ \omega_0 t) = \frac{1}{2\pi}\int_0^{2\pi} F_n^*(s)D_M(s-\omega_0 t)\ dt \quad \text{and} \quad s_M(F_n^*,\ \omega_0 t) - F_n^*(\omega_0 t)$$

$$= O\left(\omega(F_n^*,\ \frac{1}{M})\right)\log M.$$

By (4.7), we have

$$J_{M,n}(t) = \frac{2\pi}{\omega_0} s_M(F_n^*;\ \omega_0 t) \quad \text{and} \quad J_{M,n}(t) = \frac{2\pi}{\omega_0} F_n^*(\omega_0 t) + O\left(\frac{1}{\omega_0}\right)\omega\left(F_n^*;\ \frac{1}{M}\right)\log M. \tag{4.8}$$

By (4.5), (4.6), and the properties of the modulus of continuity, we get

$$\omega(F_n^*;\ \delta) \le 2\omega\left(f;\ \frac{\delta}{\omega_0}\right) \parallel W \parallel_\infty + 2\omega\left(W;\ \frac{\delta}{\omega_0}\right) \parallel f \parallel_\infty =: p_f(\delta), \qquad \text{and} \tag{4.9}$$

$$F_n^*(\omega_0 t) = F_n(\omega_0 t) = f(t)\overline{W(t - nt_0)}, \qquad t \in (a + nt_0,\ a + nt_0 + \frac{2\pi}{\omega_0}).$$

Again, by (4.8) and (4.9), we obtain that for $t \in (a + nt_0,\ a + nt_0 + \frac{2\pi}{\omega_0})$,

$$J_{M,n}(t)W(t - nt_0) = \frac{2\pi}{\omega_0} f(t)|W(t - nt_0)|^2 + O\left(\frac{1}{\omega_0}\right) \parallel W \parallel_\infty p_f\left(\frac{1}{M}\right)\log M. \tag{4.10}$$

For $t \notin \left(a + nt_0,\ a + nt_0 + \frac{2\pi}{\omega_0}\right)$, by (2.2), we have $W(t - nt_0) = 0$. So (4.10) still holds.

By (2.2), the sum $\sum\limits_n J_{M,n}(t)W(t - nt_0)$ has only $\left[\frac{b-a}{t_0}\right] + 1$ nonzero terms and $\left[\frac{b-a}{t_0}\right] + 1 \le \frac{2(b-a)}{t_0} \le \frac{4\pi}{t_0\omega_0}$. Therefore, by (4.4) and (4.10), we get

$$G(t)S_{M,N}(f;\ t) = \frac{2\pi}{\omega_0} f(t) \sum_{|n| \le N} |W(t - nt_0)|^2 + O\left(\frac{\parallel W \parallel_\infty}{t_0\omega_0^2}\right) p_f\left(\frac{1}{M}\right)\ \log M \quad (t \in \mathbb{R}). \tag{4.11}$$

Arbitrarily take $l \in \mathbb{Z}^+$. Denote $N^* = \left[\frac{l + |a| + |b|}{t_0}\right]$. When $|n| > N^*$, we have $W(t - nt_0) = 0\ (|t| \le l)$. From this and (2.3), we conclude that for $N > N^*$ and $|t| < l$, the first term of the right-hand side of (4.11) is equal to $G(t)f(t)$. Again, by (4.3) and (4.9), we obtain finally (4.2). Theorem 4.1 is proved. □

Based on Theorem 4.1, we conclude from a known result of the Fourier series [10] that

***Theorem* 4.2** — *Under the conditions of Theorem 4.1, if any one of the following conditions is satisfied*:

(*i*) $\lim\limits_{\delta \to 0}\left(\log\frac{1}{\delta}\right)\ \omega(f; \delta) = 0$ *and* $\lim\limits_{\delta \to 0}\left(\log\frac{1}{\delta}\right)\ \omega(W; \delta) = 0,$

(*ii*) f *and* W *are both continuous functions of bounded variation on* $\mathbb{R}$,

then the W-H frame series $\sum\limits_{m,n}(f,\ W_{m,n})\widetilde{W}_{m,n}(t)$ *converges uniformly to* $f(t)$ *on every compact set of* $\mathbb{R}$.

PROOF : If (i) holds, using Theorem 4.1, we obtain the desired result.

Suppose that (ii) holds. Let F_n^* be stated as in (4.5) and (4.6). By the assumption, we see that F_n^* is a $2\pi-$periodic continuous function and is of bounded variation on $[0,\ 2\pi]$. By a known result [10], we see that the partial sums of the Fourier series of F_n^* satisfy

$$\lim_{M\to\infty} s_M(F_n^*;\ t) = F_n^*(t) \tag{4.13}$$

uniformly on $\mathbb{R}$. Using the argument of Theorem 4.1, we know that for any $l > 0$, there is a $N^* > 0$, which only depends on l, such that for any $N > N^*$,

$$G(t)S_{M,N}(f;\ t) = \frac{2\pi}{\omega_0} \sum_{|n|\le N} s_M(F_n^*;\ \omega_0 t)W(t-nt_0) =:\ I_M^{(1)}(t) + I_M^{(2)}(t) \quad (|t|\le l),$$

where G is stated in (2.3) and

$$I_M^{(1)}(t) = \frac{2\pi}{\omega_0} \sum_{|n|\le N} (s_M(F_n^*;\ \omega_0 t) - F_n^*(\omega_0 t))W(t-nt_0),$$

$$I_M^{(2)}(t) = \frac{2\pi}{\omega_0} \sum_{|n|\le N} F_n^*(\omega_0 t)W(t-nt_0) = f(t)G(t).$$

From this and (4.13) and $\min\limits_{t\in\mathbb{R}} G(t) \ge \frac{2\pi}{\omega_0} m^* > 0$, we obtain the desired result. Theorem 4.2 is proved. □

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