Volume 9,Number 4 October 2007
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.9,NO.4,437-448,2007,COPYRIGHT 2007 EUDOXUS PRESS ,LLC

Continuity of Fourier Transforms of Band-Limited Wavelets

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Abstract

Based on the research of the supports of dimension functions for band-limited wavelets, we show that if the Fourier transform of a band-limited wavelet is continuous at every boundary point of the support of its Fourier transform, then this band-limited wavelet is associated with a multiresolution analysis.

Keywords: multiresolution analysis; band-limited wavelet; support

MSC: 42C40

1. Introduction

If a wavelet is associated with a multiresolution analysis(MRA), one call it an MRA wavelet, otherwise, one call it a non-MRA wavelet. It is well-known that if a wavelet has a compact support, it must be an MRA wavelet[7,p363]. However if a wavelet is band-limited, it may not be an MRA wavelet. Journe [7], Bownik[3], and Behera[2] successively constructed many band-limited non-MRA wavelets. The purpose of the present paper is to study that under what conditions, a band-limited wavelet is an MRA wavelet. In this field, there are the following results in literature.

Proposition 1.1[7, p364]. *If ψ is a band-limited wavelet such that $|\widehat{\psi}|$ is continuous, then ψ is an MRA wavelet.*

Proposition 1.2[4,6, p338]. *If a wavelet ψ is such that* $\mathrm{supp}\widehat{\psi} \subset [-\frac{8}{3}\alpha,\ 4\pi - \frac{4}{3}\alpha]$ $(0 < \alpha \leq \pi)$, *then ψ is an MRA wavelet.*

Based on the study of the supports of dimension functions of band-limited wavelets, we improve Proposition 1.1. We show that " a band-limited wavelet ψ is an MRA wavelet provided that $|\widehat{\psi}|$ is continuous at every boundary point of $\mathrm{supp}\widehat{\psi}$ ".

2. MRA wavelets and dimension functions

We first recall some basis notions.

Let $\{V_m\}$ be a sequence of closed subspaces of $L^2(R)$. If it satisfies the following conditions:

(i) $V_m \subset V_{m+1}$ $(m \in Z)$, $\overline{\bigcup_m V_m} = L^2(R)$, $\bigcap_m V_m = \{0\}$,

(ii) $f \in V_m \leftrightarrow f(2\cdot) \in V_{m+1}$ $(m \in Z)$,

(iii) there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - n),\ n \in Z\}$ is an orthonormal basis of V_0,

then $\{V_m\}$ is called a multiresolution analysis (MRA) and φ is called a scaling function[5,8].

Let $\psi \subset L^2(R)$ and the system

$$\{2^{\frac{m}{2}}\psi(2^m\cdot - n),\quad m \in Z,\ n \in Z\}$$

be an orthonormal basis of $L^2(R)$. Then ψ is said to be a wavelet.

Let ψ be a wavelet. For $m \in Z$, let W_m be the closure in $L^2(R)$ of the span $\{2^{\frac{m}{2}}\psi(2^m\cdot - n) :\ n \in Z\}$ and $V_m = \bigoplus_{l=-\infty}^{m-1} W_l$, where $\bigoplus$ is the orthogonal sum. If $\{V_m\}$ is an MRA, then ψ is said to be an MRA wavelet[7,p355], otherwise it is said to be a non-MRA wavelet.

For a wavelet ψ, define its dimension function $D_\psi(\omega)$[7] as follows:

$$D_\psi(\omega) = \sum_{m=1}^{\infty}\sum_{n\in Z} |\widehat{\psi}(2^m(\omega + 2n\pi))|^2. \tag{2.1}$$

Hereafter $\widehat{f}$ is the Fourier transform of $f \in L^2(R)$.

Proposition 2.1[1,7,p360]. *The dimension function $D_\psi(\omega)$ of a wavelet ψ is an integer valued function.*

The dimension functions can characterize MRA wavelets.

Proposition 2.2[7,p363]. *Let ψ be a wavelet. Then the following statements are equivalent*:

(i) *ψ is an MRA wavelet* (ii)$D_\psi(\omega) = 1$ *a.e.* (iii) $D_\psi(\omega) > 0$ *a.e.*.

Let f be a complex valued function on R. The closure of the point set $\{\omega \in R;\ f(\omega) \neq 0)\}$ is said to be the support of f[9, p38]. If supp$\widehat{f}$ is bounded, then f is said to be band-limited.

Remark 2.3. *In general, $E = \text{supp} f$ can not imply $f(\omega) \neq 0$ a.e. $\omega \in E$.*

Throughout this paper, ∂E, E^o, and $\overline{E}$ denote the boundary, the interior, and the closure of a point set $E \subset R$, respectively. $|E|$ expresses the Lebesgue measure of E. For $a,\ b \in R$,

$$E + a = \{\omega + a,\ \omega \in E\}, \quad bE = \{b\omega,\ \omega \in E\}, \quad \text{and} \quad 2\pi Z = \{2\pi n,\ n \in Z\}.$$

3. Main theorems

Theorem 3.1. *Let ψ be a band-limited wavelet with* $\text{supp}\widehat{\psi} = G$. *Suppose that $\widehat{\psi}(\omega) \neq 0$ a.e. $\omega \in G$ and $|\widehat{\psi}|$ is continuous at every point on the boundary ∂G. Then ψ is an MRA wavelet.*

From Remark 2.3, we see that in general, $\text{supp}\widehat{\psi} = G$ can not imply $\widehat{\psi}(\omega) \neq 0$ *a.e.* $\omega \in G$. Theorem 3.1 improves Proposition 1.1 and is a corollary of the following theorem.

Theorem 3.2. *Let ψ be a band-limited wavelet with* $\text{supp}\widehat{\psi} = G$. *Denote $\Omega = \bigcup\limits_{m \geq 0} 2^{-m}G$. Suppose that $\widehat{\psi}(\omega) \neq 0$ a.e. $\omega \in G$ and $|\widehat{\psi}|$ is continuous at every point on $\partial G \bigcap \partial\Omega$. Then ψ is an MRA wavelet.*

The set $\partial G \bigcap \partial\Omega$ is only a subset of the boundary ∂G. For a Meyer wavelet ψ[5, p117],

$$G = \text{supp}\widehat{\psi} = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \bigcup [\frac{2\pi}{3}, \frac{8\pi}{3}], \qquad \Omega = [-\frac{8\pi}{3}, \frac{8\pi}{3}] \setminus \{0\}.$$

So

$$\partial\Omega = \{-\frac{8\pi}{3},\ 0,\ \frac{8\pi}{3}\}, \qquad \partial G = \{-\frac{8\pi}{3},\ -\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{8\pi}{3}\}$$

and then $\partial G \bigcap \partial\Omega = \{-\frac{8\pi}{3},\ \frac{8\pi}{3}\}$.

4. Proof of theorem 3.2

We only prove Theorem 3.2 since Theorem 3.1 is a corollary of Theorem 3.2.

Let ψ be a band-limited wavelet. Then by (2.1), the dimension function D_ψ can be written in the form

$$D_\psi(\omega) = \sum_{n \in Z} g(\omega + 2n\pi), \quad \text{where} \quad g(\omega) = \sum_{m=1}^{\infty} |\widehat{\psi}(2^m\omega)|^2. \tag{4.1}$$

Denote

$$G = \text{supp}\widehat{\psi} \quad \text{and} \quad \Omega = \bigcup_{m \geq 0} 2^{-m}G. \tag{4.2}$$

Since ψ is band-limited, G is bounded, further Ω is bounded. From (4.1) and (4.2), and noticing that $g(\omega) \geq 0$ *a.e.*, we get

$$\mathrm{supp} g = \overline{\bigcup_{m\geq 1} 2^{-m}G} = \frac{1}{2}\overline{\Omega}. \tag{4.3}$$

and

$$\mathrm{supp} D_\psi = \overline{\bigcup_n} \mathrm{supp} g(\cdot + 2n\pi) = \overline{\bigcup_n} (\frac{1}{2}\overline{\Omega} - 2n\pi). \tag{4.4}$$

Hereafter, $\bigcup\limits_n = \bigcup\limits_{n\in Z}$.

Lemma 4.1. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi} = G$*. Suppose that* $\widehat{\psi}(\omega) \neq 0$ *a.e.* $\omega \in G$*. Then*

(i) $\mathrm{supp} g = \frac{1}{2}\Omega \bigcup \{0\}$ *and* $g(\omega) > 0$ *a.e.* $\omega \in \frac{1}{2}\Omega$

(ii) $\mathrm{supp} D_\psi = \left(\bigcup\limits_n (\frac{1}{2}\Omega - 2n\pi)\right) \bigcup 2\pi Z$ *and* $D_\psi(\omega) > 0$ *a.e.* $\omega \in \mathrm{supp} D_\psi$

(iii) *If ψ is a non-MRA wavelet, then*

$$|R \setminus \mathrm{supp} D_\psi| > 0, \tag{4.5}$$

where g, D_ψ and Ω are stated in (4.1) *and* (4.2), *and* $2\pi Z = \{2\pi n,\ n \in Z\}$.

Proof: (i) We first prove that $\overline{\Omega} \subset \Omega \bigcup \{0\}$.

Let $\omega_0 \neq 0$ and $\omega_0 \in \overline{\Omega}$. Take $r > 0$ such that $0 \notin (\omega_0 - r, \omega_0 + r)$. Since G is bounded, there exists a $m_1 \in Z^+$ such that $(\omega_0 - r, \omega_0 + r) \bigcap 2^{-m}G = \emptyset \quad (m > m_1)$. So

$$\omega_0 \notin \overline{\bigcup_{m>m_1} 2^{-m}G}. \tag{4.6}$$

Since G is closed, by (4.2), noticing that the union of the finitely many closed set is a closed set, we have

$$\overline{\Omega} = \left(\bigcup_{0\leq m\leq m_1} 2^{-m}G \right) \bigcup \left(\overline{\bigcup_{m>m_1} 2^{-m}G} \right).$$

Again noticing that $\omega_0 \in \overline{\Omega}$ and (4.6), we obtain that

$$\omega_0 \in \bigcup_{0\leq m\leq m_1} 2^{-m}G \subset \Omega \quad \text{(by (4.2))},$$

and so $\overline{\Omega} \subset \Omega \bigcup \{0\}$.

Take $\omega_0 \in G$. By (4.2), we see that $\{2^{-m}\omega_0\}_0^\infty \subset \Omega$. Since $2^{-m}\omega_0 \to 0$ $(m \to \infty)$, we get $0 \in \overline{\Omega}$. So $\overline{\Omega} \supset \Omega \bigcup \{0\}$. Hence, we have

$$\overline{\Omega} = \Omega \bigcup \{0\}. \tag{4.7}$$

From this and (4.3), we get $\mathrm{supp} g = \frac{1}{2}\Omega \bigcup \{0\}$.

Since $\widehat{\psi}(\omega) \neq 0$ $a.e.\, \omega \in G$, we see that $|\widehat{\psi}(2^m\omega)| > 0 \quad a.e.\ \omega \in 2^{-m}G$. Therefore, by (4.1) and (4.2),

$$g(\omega) > 0 \ \ a.e.\ \omega \in \bigcup_{m \geq 1} 2^{-m}G = \frac{1}{2}\Omega.$$

(i) is proved.

(ii) Denote

$$Q = \bigcup_n (\frac{1}{2}\overline{\Omega} - 2n\pi). \tag{4.8}$$

We will prove that Q is a closed set.

Let $\zeta \in \overline{Q}$. If ζ is an isolated point of $\overline{Q}$, then $\zeta \in Q$. If ζ is not an isolated point of $\overline{Q}$, then there exists a sequence $\{\zeta_l\} \subset Q$ and $\zeta_l \to \zeta$. Take $\delta > 0$. Since $\zeta_l \to \zeta$ and $\frac{1}{2}\overline{\Omega}$ is bounded, we can find a $N > 0$ such that

$$\{\zeta_l\}_{l>N} \subset (\zeta - \delta, \zeta + \delta) \quad \text{and} \quad (\frac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta) = \emptyset \quad (|n| > N).$$

From this, and $\{\zeta_l\} \subset Q$ and (4.8), we have

$$\begin{aligned} \{\zeta_l\}_{l>N} \subset (\ Q \bigcap (\zeta - \delta, \zeta + \delta)\ ) &= \bigcup_n \big(\ (\tfrac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta)\ \big) \\ &= \bigcup_{|n| \leq N} \big(\ (\tfrac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta)\ \big) \\ &\subset \bigcup_{|n| \leq N} (\tfrac{1}{2}\overline{\Omega} - 2n\pi) =: Q_N. \end{aligned}$$

Since Q_N is closed, from $\zeta_l \to \zeta$, we see that $\zeta \in Q_N \subset Q$. Hence Q is a closed set.

From this and (4.4), and (4.8), it follows that

$$\mathrm{supp} D_\psi = \overline{Q} = Q = \bigcup_n (\frac{1}{2}\overline{\Omega} - 2n\pi).$$

Again by (4.7), we have

$$\mathrm{supp} D_\psi = \left(\bigcup_n (\frac{1}{2}\Omega - 2n\pi\,) \right) \bigcup 2\pi Z. \tag{4.9}$$

By (i), we get

$$g(\omega+2n\pi)>0\ a.e.\omega\in(\frac{1}{2}\Omega-2n\pi)\quad \text{for any } n\in Z.$$

Again noticing that $g(\omega)\geq 0\ a.e.\,\omega\in R$, by (4.1), and (4.9), we get $D_\psi(\omega)>0\ a.e.\,\omega\in \mathrm{supp}D_\psi$. (ii) follows.

(iii) If (4.5) is not valid, then $|R\setminus \mathrm{supp}D_\psi|=0$. Since $R\setminus\mathrm{supp}D_\psi$ is an open set, we have

$$R\setminus\mathrm{supp}D_\psi=\emptyset,\quad \text{i.e.}\quad \mathrm{supp}D_\psi=R.$$

By (ii), we get $D_\psi(\omega)>0\ a.e.\,\omega\in R$. From Proposition 2.2, it follows that ψ is an MRA wavelet. This is contrary to the assumption in (iii). So we get (iii). Lemma 4.1 is proved.

Lemma 4.2. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi}=G$. *Suppose that $\widehat{\psi}(\omega)\neq 0\ a.e.\ \omega\in G$ and $|\widehat{\psi}|$ is continuous at every point on $\partial G\bigcap\partial\Omega$. Then the function g is continuous and vanishes at every point of $\partial(\frac{1}{2}\Omega)\setminus\{0\}$, where g and Ω are stated in* (4.1) *and* (4.2).

Proof: Let $\omega_0\in\partial(\frac{1}{2}\Omega)\setminus\{0\}$. Take $r>0$ such that $0\notin(\omega_0-r,\omega_0+r)$.

Since G is bounded, there exists $m_1>0$ such that

$$(\omega_0-r,\omega_0+r)\bigcap 2^{-m}G=\emptyset\quad (m>m_1).$$

Again since $\mathrm{supp}\widehat{\psi}(2^m\cdot)=2^{-m}G$, we see that

$$\widehat{\psi}(2^m\omega)=0,\quad \omega\in(\omega_0-r,\omega_0+r)\qquad (m>m_1).$$

By (4.1), we get

$$g(\omega)=\sum_{m=1}^{m_1}|\widehat{\psi}(2^m\omega)|^2,\quad \omega\in(\omega_0-r,\omega_0+r). \tag{4.10}$$

Since $\omega_0\in\partial(\frac{1}{2}\Omega)$ and $\frac{1}{2}\Omega=\bigcup\limits_{m\geq 1}2^{-m}G$ (by (4.2)), we see that for any $m\geq 1$, ω_0 is not an inner point of $2^{-m}G$, otherwise, $\omega_0\in(\frac{1}{2}\Omega)^o$, this is contrary to $\omega_0\in\partial(\frac{1}{2}\Omega)$. Hence for any $m\geq 1$,

$$\omega_0\notin 2^{-m}G\quad \text{or}\quad \omega_0\in\partial(2^{-m}G).$$

(i) In the case of $\omega_0\notin 2^{-m}G$. Since $\mathrm{supp}|\widehat{\psi}(2^m\cdot)|=2^{-m}G$ and $\omega_0\notin 2^{-m}G$, we have $\omega_0\notin\mathrm{supp}|\widehat{\psi}(2^m\cdot)|$. So $|\widehat{\psi}(2^m\cdot)|$ is continuous and vanishes at ω_0.

(ii) In the case of $\omega_0 \in \partial(2^{-m}G)$, we will prove that $|\widehat{\psi}(2^m\cdot)|$ is also continuous and vanishes at ω_0. From $\omega_0 \in \partial(2^{-m}G)$, we have $2^m\omega_0 \in \partial G$. Now we first prove that

$$\omega_0 \in \partial(2^{-m}\Omega). \tag{4.11}$$

If (4.11) is not true, then $2^m\omega_0 \notin \partial\Omega$. From this and $2^m\omega_0 \in \partial G \subset G \subset \Omega$, it follows that $2^m\omega_0 \in \Omega^o$. So there exists a $\epsilon > 0$ such that $(2^m\omega_0 - \epsilon,\ 2^m\omega_0 + \epsilon) \subset \Omega$, i.e.

$$(\omega_0 - 2^{-m}\epsilon,\ \omega_0 + 2^{-m}\epsilon) \subset 2^{-m}\Omega.$$

By $m \geq 1$ and (4.2), we have

$$2^{-m}\Omega = \bigcup_{k\geq m} 2^{-k}G \subset \bigcup_{k\geq 1} 2^{-k}G = \frac{1}{2}\Omega.$$

So

$$(\omega_0 - 2^{-m}\epsilon, \omega_0 + 2^{-m}\epsilon) \subset \frac{1}{2}\Omega, \qquad \text{i.e.} \quad \omega_0 \in (\frac{1}{2}\Omega)^o.$$

This is contrary to $\omega_0 \in \partial(\frac{1}{2}\Omega) \setminus \{0\}$. So (4.11) holds.

From $\omega_0 \in \partial(2^{-m}G)$ and (4.11), we have

$$\omega_0 \in (\partial(2^{-m}G) \bigcap \partial(2^{-m}\Omega)) = 2^{-m}(\partial G \bigcap \partial\Omega). \tag{4.12}$$

By the assumption: $|\widehat{\psi}|$ is continuous at every point on $\partial G \bigcap \partial\Omega$ and (4.12): $\omega_0 \in 2^{-m}(\partial G \bigcap \partial\Omega)$, we conclude that $|\widehat{\psi}(2^m\cdot)|$ is continuous at ω_0.

By $\widehat{\psi}(\omega) = 0$ $(\omega \notin G)$, we have $\widehat{\psi}(2^m\omega) = 0$ $(\omega \notin 2^{-m}G)$. Again since $\omega_0 \in \partial(2^{-m}G)$, there exists a sequence $\{\zeta_l\}_1^\infty$ such that

$$\zeta_l \to \omega_0 \ (l \to \infty) \quad \text{and} \quad \widehat{\psi}(2^m\zeta_l) = 0 \quad \text{for all } l.$$

Since $|\widehat{\psi}(2^m\cdot)|$ is continuous at ω_0, we get $\widehat{\psi}(2^m\omega_0) = 0$.

From (i) and (ii), we see that for $m \geq 1$, $|\widehat{\psi}(2^m\cdot)|$ is continuous and vanishes at ω_0. Again by (4.10), we obtain that g is continuous and vanishes at ω_0. Lemma 4.2 is proved.

Lemma 4.3. *Let ψ be a band-limited non-MRA wavelet with* $\text{supp}\widehat{\psi} = G$. *If* $\widehat{\psi}(\omega) \neq 0$ *a.e.* $\omega \in G$, *then there exists a point* $\omega^* \in (0,\ 2\pi)$ *such that for an arbitrarily small* $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \text{supp}D_\psi| > 0 \quad \text{and} \quad |(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \text{supp}D_\psi| > 0,$$

where D_ψ *is stated in* (4.1).

Proof: Since ψ is a non-MRA wavelet, by Lemma 4.1(iii), we see that $|R \setminus \text{supp}D_\psi| > 0$. By (4.4) and (4.2), we see that $|\text{supp}D_\psi| > |\frac{1}{2}\Omega| > |\frac{1}{2}G| > 0$. By Lemma 4.1(ii), we see that $\text{supp}D_\psi + 2n\pi = \text{supp}D_\psi$ $(n \in Z)$. Therefore, we have

$$|(0, 2\pi) \bigcap \text{supp}D_\psi| > 0 \quad \text{and} \quad |(0, 2\pi) \setminus \text{supp}D_\psi| > 0.$$

Hence there exists $\eta > 0$ such that

$$|(\eta, 2\pi - \eta) \bigcap \text{supp}D_\psi| > 0 \quad \text{and} \quad |(\eta, 2\pi - \eta) \setminus \text{supp}D_\psi| > 0. \tag{4.13}$$

Since $\text{supp}D_\psi$ is a closed set, $(\eta, 2\pi - \eta) \setminus \text{supp}D_\psi$ is an open set. So there exist $\omega_1 \in (\eta, 2\pi - \eta)$ and $\epsilon_1 > 0$ such that $(\omega_1 - \epsilon_1, \omega_1 + \epsilon_1) \subset ((\eta, 2\pi - \eta) \setminus \text{supp}D_\psi)$, so we have

$$|(\omega_1 - \epsilon_1,\ \omega_1 + \epsilon_1) \setminus \text{supp}D_\psi| = 2\epsilon_1. \tag{4.14}$$

Below we prove that there exists a $\omega_2 \in [\eta, 2\pi - \eta]$ such that for an arbitrarily small $\epsilon > 0$,

$$|(\omega_2 - \epsilon, \omega_2 + \epsilon) \bigcap \text{supp}D_\psi| > 0. \tag{4.15}$$

If it is not true, then for any $\omega \in [\eta, 2\pi - \eta]$, there exists some $\epsilon_\omega > 0$ such that

$$|(\omega - \epsilon_\omega, \omega + \epsilon_\omega) \bigcap \text{supp}D_\psi| = 0. \tag{4.16}$$

Since $(\bigcup_{\omega \in [\eta, 2\pi - \eta]} (\omega - \epsilon_\omega, \omega + \epsilon_\omega)) \supset [\eta, 2\pi - \eta]$, using the theorem of finite covering, we know that there are finitely many points $\{\tau_l\}_1^s$ in the closed interval $[\eta, 2\pi - \eta]$ such that

$$\left(\bigcup_{l=1}^{s} (\tau_l - \epsilon_{\tau_l}, \tau_l + \epsilon_{\tau_l}) \right) \supset [\eta, 2\pi - \eta].$$

Thus,

$$\left(\bigcup_{l=1}^{s}\left((\tau_l-\epsilon_{\tau_l},\tau_1+\epsilon_{\tau_l})\bigcap \mathrm{supp}D_\psi\right)\right)\supset([\eta,2\pi-\eta]\bigcap \mathrm{supp}D_\psi).$$

Again by (4.16), we get

$$|[\eta,2\pi-\eta]\bigcap \mathrm{supp}D_\psi|\le\sum_{l=1}^{s}|(\tau_l-\epsilon_{\tau_l},\tau+\epsilon_{\tau_l})\bigcap \mathrm{supp}D_\psi|=0.$$

This is contrary to the first formula of (4.13), so (4.15) holds.

If, for an arbitrarily small $\epsilon>0$, the following inequality

$$|(\omega_2-\epsilon,\omega_2+\epsilon)\setminus \mathrm{supp}D_\psi|>0 \tag{4.17}$$

holds, combining (4.17) with (4.15), then we see that the point ω_2 is just a desired point ω^*.

If, for some $\epsilon_2>0$, (4.17) does not hold, then $|(\omega_2-\epsilon_2,\omega_2+\epsilon_2)\setminus \mathrm{supp}D_\psi|=0$. From this and (4.14), we have

$$|(\omega_2-\epsilon_2,\omega_2+\epsilon_2)\bigcap \mathrm{supp}D_\psi|=2\epsilon_2 \quad\text{and}\quad |(\omega_1-\epsilon_1,\omega_1+\epsilon_1)\setminus \mathrm{supp}D_\psi|=2\epsilon_1. \tag{4.18}$$

From (4.18), we see that $\omega_1\ne\omega_2$. Without loss of generality, we assume that $\omega_1<\omega_2$.

Below we show that in the interval (ω_1,ω_2), there exists a desired point ω^*.

Define a point set E as

$$E=\{\zeta\in(\omega_1,\omega_2):\quad |(\zeta-r,\zeta+r)\bigcap \mathrm{supp}D_\psi|=2r\quad\text{for an arbitrarily small}\ \ r>0\}. \tag{4.19}$$

Clearly, E is an open set. By (4.18) and (4.19), it is easy to see that there exists $\delta>0$ such that

$$(\omega_1,\omega_1+\delta)\subset((\omega_1,\omega_2)\setminus E)\quad\text{and}\quad(\omega_2-\delta,\ \omega_2)\subset E. \tag{4.20}$$

So $\omega_1\notin\overline{E}$ and $\overline{E}$ is a nonempty, closed set, and then there exists a point $\omega^*\in\overline{E}$ such that

$$|\omega_1-\omega^*|=\inf_{\omega\in\overline{E}}|\omega_1-\omega|.$$

From this and (4.20), noticing that E is an open set, we see that the point ω^* possesses the following properties:

$$\omega^*\in\overline{E},\quad \omega^*\notin E\qquad\text{and}\qquad \omega^*\in(\omega_1,\omega_2). \tag{4.21}$$

Now we prove that the point ω^* is a desired point.

First, it is easy to see that

$$\omega^* \subset (\omega_1, \omega_2) \subset [\eta, 2\pi - \eta] \subset (0, 2\pi). \tag{4.22}$$

From (4.21), it follows that there exists $\{\zeta_n\} \subset E$ such that $\zeta_n \to \omega^*$. So, for an arbitrarily small $\epsilon > 0$, there exists some ζ_{n_0} in $\{\zeta_n\}$ such that $\zeta_{n_0} \in (\omega^* - \epsilon, \omega^* + \epsilon)$. Again since $\zeta_{n_0} \in E$, by (4.19), there exists $r > 0$ such that

$$|(\zeta_{n_0} - r, \zeta_{n_0} + r) \bigcap \mathrm{supp} D_\psi| = 2r, \qquad (\zeta_{n_0} - r, \zeta_{n_0} + r) \subset (\omega^* - \epsilon, \omega^* + \epsilon)$$

simultaneously hold. From this, it follows that for an arbitrarily small $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| > 0. \tag{4.23}$$

On the other hand, by (4.21), we have $\omega^* \in (\omega_1, \omega_2) \setminus E$. Again by (4.19), we see that for an arbitrarily small $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| < 2\epsilon.$$

Noticing that $|(\omega^* - \epsilon, \omega^* + \epsilon)| = 2\epsilon$, we have

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \mathrm{supp} D_\psi| > 0. \tag{4.24}$$

Combining (4.22), (4.23) with (4.24), we see that the point ω^* is just a desired point. Lemma 4.3 is proved.

Proof of theorem 3.2: We will give a proof by contradiction.

Suppose that ψ is a non-MRA wavelet. Then by Lemma 4.3, there exists a point $\omega^* \in (0, 2\pi)$ such that for an arbitrarily small $\epsilon > 0$, the following two formulas hold simultaneously,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| > 0 \quad \text{and} \quad |(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \mathrm{supp} D_\psi| > 0. \tag{4.25}$$

This implies $\omega^* \in \partial(\mathrm{supp} D_\psi)$. By Lemma 4.1(ii), we see that for any n, the point ω^* is not an inner point of

$\frac{1}{2}\Omega - 2n\pi$, otherwise, ω^* is an inner point of suppD_ψ, this is contrary to $\omega^* \in \partial(\text{supp}D_\psi)$. So, for any n,

$$\omega^* \notin (\frac{1}{2}\Omega - 2n\pi) \quad \text{or} \quad \omega^* \in \partial(\frac{1}{2}\Omega - 2n\pi).$$

In the case of $\omega^* \notin (\frac{1}{2}\Omega - 2n\pi)$. Noticing that $\omega^* \in (0, 2\pi)$, we have $\omega^* + 2n\pi \neq 0$, so $\omega^* + 2n\pi \notin (\frac{1}{2}\Omega \bigcup \{0\})$. By Lemma 4.1(i), $\omega^* + 2n\pi \notin \text{supp}g$. So $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*

In the case of $\omega^* \in \partial(\frac{1}{2}\Omega - 2n\pi)$. Since $\omega^* + 2n\pi \neq 0$, we have $\omega^* + 2n\pi \in \partial(\frac{1}{2}\Omega) \setminus \{0\}$. By Lemma 4.2, $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*.

From this, we know that for any n, $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*. Since $\text{supp}g = \frac{1}{2}\Omega \bigcup \{0\}$ is bounded, for $\epsilon > 0$, there exists a $N > 0$ such that

$$(\omega^* - \epsilon,\ \omega^* + \epsilon) \bigcap \text{supp}g(\cdot + 2n\pi) = \emptyset \quad (|n| > N).$$

So the series $\sum\limits_{n \in Z} g(\omega + 2n\pi)$ has only finitely many nonzero terms in the neighborhood $(\omega^* - \epsilon, \omega^* + \epsilon)$. By (4.1), $D_\psi(\omega)$ is also continuous and vanishes at $\omega = \omega^*$. So there exists a $\eta > 0$

$$D_\psi(\omega) < \frac{1}{2}, \quad \omega \in (\omega^* - \eta, \omega^* + \eta).$$

From this and Lemma 4.1(ii),

$$0 < D_\psi(\omega) < \frac{1}{2} \quad a.e. \quad \omega \in ((\omega^* - \eta, \omega^* + \eta) \bigcap \text{supp}D_\psi). \tag{4.26}$$

By (4.25), we get

$$|(\omega^* - \eta, \omega^* + \eta) \bigcap \text{supp}D_\psi| > 0.$$

From this and (4.26), we see that the formula $0 < D_\psi(\omega) < \frac{1}{2}$ holds on a point set with positive measure. However, Proposition 2.1 shows that $D_\psi(\omega)$ is an integer valued function, this is a contradiction. Hence ψ is an MRA wavelet. Theorem 3.2 is proved.

Since the set $\partial G \bigcap \partial\Omega$ is a subset of the boundary ∂G, using Theorem 3.2, we obtain Theorem 3.1 immediately.

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