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Appl. Comput. Harmon. Anal. 22 (2007) 141–156

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Supports of Fourier transforms of scaling functions

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Received 23 November 2004; revised 28 April 2006; accepted 23 May 2006

Available online 3 July 2006

Communicated by Matthias Holschneider

Abstract

We characterize the support G of the Fourier transform of the band-limited scaling function and give an approach to the construction of the scaling function. Based on the relations of translates of three point sets G, $\frac{1}{2}G$ and $G \setminus \frac{1}{2}G$, we reveal an essential difference of scaling functions corresponding to MRA, weakly translation invariant MRA and translation invariant MRA. Moreover, we consider the case that support of the Fourier transform of the scaling function is a convex set. Finally, we show that the support of Fourier transform of any scaling function cannot be a closed ball.

Keywords: Multiresolution analysis; Scaling function; Support

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1. Introduction

In 1988, Mallat introduced a concept of multiresolution analysis (MRA) which is a fundamental concept in wavelet analysis.

Definition 1.1. [1,2,6] Let $\{V_m\}_{m\in Z}$ be a sequence of closed subspaces of $L^2(R^d)$ satisfying:

(a) $V_m \subset V_{m+1}$ $(m \in Z)$, $\overline{\bigcup_m V_m} = L^2(R^d)$, $\bigcap_m V_m = \{0\}$.
(b) $f \in V_m$ if and only if $f(2\cdot) \in V_{m+1}$ $(m \in Z)$.
(c) There exists a $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}_{n\in Z^d}$ is an orthonormal basis of V_0.

Then $\{V_m\}$ is called an MRA and φ a scaling function.

In 1991, Madych introduced the concept of translation invariant MRAs.

Definition 1.2. [5] Let $\{V_m\}$ be an MRA. If for any $f \in V_0$ and $\alpha \in R^d$, we have $f(t-\alpha) \in V_0$, then $\{V_m\}$ is called a translation invariant MRA and the corresponding scaling function is called an M-type scaling function.

Replacing $f(t-\alpha) \in V_0$ by $f(t-\alpha) \in V_1$, Walter extended the above concept in 1994.

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doi:10.1016/j.acha.2006.05.007

Definition 1.3. [7,8] Let $\{V_m\}$ be an MRA. If for any $f \in V_0$ and $\alpha \in R^d$, we have $f(t-\alpha) \in V_1$, then $\{V_m\}$ is called a weakly translation invariant MRA and the corresponding scaling function is called a W-type scaling function.

One easily see that if φ is an M-type scaling function, then φ must be a W-type scaling function and if φ is a W-type scaling function, then φ must be a scaling function. Conversely, it is not true. For example, Meyer's scaling function is a W-type scaling function, but it is not an M-type scaling function [3,7]. Any scaling function φ with $\operatorname{supp}\hat{\varphi} = R^d$ is not a W-type scaling function. Up to now there is not a function available which is a band-limited scaling function but is not a W-type scaling function, we present such an example (Example 2.10 of this paper).

Madych [5] showed that there exists an M-type scaling function φ with $\operatorname{supp}\hat{\varphi} = G$ if and only if

$$\text{(i)}\quad G \subset 2G, \qquad \text{(ii)}\quad \bigcup_m 2^m G \simeq R^d, \qquad \text{(iii)}\quad G + 2\pi Z^d \simeq R^d,$$

$$\text{(iv)}\quad G \cap (G + 2\pi\nu) \simeq \emptyset \quad \bigl(\nu \in Z^d \setminus \{0\}\bigr)$$

hold. In this case, $|\hat{\varphi}| = X_G$, where X_G is the characteristic function of G.

Walter [7] showed that if φ is a W-type scaling function with $\operatorname{supp}\hat{\varphi} = G$, then (i)–(iii) and (v) hold, where

$$\text{(v)}\quad \frac{1}{2}G \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad \bigl(\nu \in Z^d \setminus \{0\}\bigr).$$

Remark 1.4. From the proofs of Madych's and Walter's theorems [5,7], one see clearly that in order to obtain the above conclusions (i)–(iii), they only use the condition that φ is a scaling function. So if φ is a scaling function, then (i)–(iii) hold.

In this paper, we will present the necessary and sufficient conditions for G to be the support of the Fourier transform of a scaling function or a W-type scaling function. Our *main results* are as follows:

(a) Let G be a bounded closed set in R^d. Then there exists a scaling function φ with $\operatorname{supp}\hat{\varphi} = G$ if and only if (i)–(iii) and (vi) hold, where

$$\text{(vi)}\quad \left(G \Big\backslash \frac{1}{2}G\right) \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad \bigl(\nu \in Z^d\bigr) \text{ (see Theorem 2.1).}$$

(b) Let G be a bounded closed set in R^d. Then there exists a W-type scaling function φ with $\operatorname{supp}\hat{\varphi} = G$ if and only if (i)–(iii) and (vii) hold, where

$$\text{(vii)}\quad G \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad \bigl(\nu \in Z^d \setminus \{0\}\bigr) \text{ (see Theorem 3.1).}$$

Combining Madych's result with our results, we see clearly that (vi), (vii) and (iv) reflect the essential difference of three kinds of scaling functions.

In the proof of Theorem 2.1, we give an approach to the constructions of the general band-limited scaling functions, and in Example 2.10, we use this method to construct a band-limited scaling function which is not a W-type scaling function.

In addition, in Section 3, we further discuss the case that the support of the Fourier transform of the scaling function is convex set (Theorem 3.3). For a band-limited scaling function φ, when the support of $\hat{\varphi}$ is convex and completely symmetric about the origin, we derive a formula:

$$[-\pi,\pi]^d \subset \operatorname{supp}\hat{\varphi} \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d \quad \text{(Theorem 3.5)}$$

and this result cannot be improved. We also discuss the relation between the support of the Fourier transform of the scaling function and the origin.

Finally, we find that any closed ball in R^d $(d > 1)$, whether its center is the origin or not, cannot be the support of the Fourier transform of some scaling function (Theorem 4.1).

Throughout this paper, we need the following notations and definitions about the point sets in R^d.

Let E be a point set in R^d. By ∂E denote the boundary of E, by E^o denote the interior of E and by $|E|$ denote the measure of E. For the point sets E and F in R^d, $E \simeq F$ means $|E \setminus F| = 0$ and $|F \setminus E| = 0$, and $E + h = \{\omega + h,\ \omega \in E\}$ $(h \in R^d)$, $kE = \{k\omega,\ \omega \in E\}$ $(k > 0)$, $E + 2\pi Z^d = \bigcup_{\nu \in Z^d}(E + 2\pi\nu)$.

If $E + 2\pi Z^d = E$, then we say that E is a 2π-periodic point set. If for any $a, b \in E$, $a + \lambda(b-a) \in E$ $(0 \leqslant \lambda \leqslant 1)$, then we say E is a convex set. If for any $\omega = (\omega_1, \ldots, \omega_d) \in E$, the 2^d points $(\pm\omega_1, \ldots, \pm\omega_d)$ all belong to E, then we say that E is completely symmetric about the origin. For a bounded convex set E, clearly, $|\partial E| = 0$ and any straight line through an interior point of E only intersects ∂E at two points.

By $D(a, \delta)$ denote a ball with center a and radius δ. By T^d denote the cube $[-\pi, \pi]^d$ and by $\{0, 1\}^d$ denote all vertices of $[0, 1]^d$. X_E is the characteristic function of E.

By $\hat{f}$ denote the Fourier transform of $f \in L^2(R^d)$. If

$$G := \operatorname{clos}\{\omega \in R^d,\ \hat{f}(\omega) \neq 0\} \quad \text{and} \quad \hat{f}(\omega) \neq 0 \quad \text{for a.e. } \omega \in G,$$

then G is called the support of $\hat{f}$ and write $G = \operatorname{supp} \hat{f}$. If $\operatorname{supp} \hat{f}$ is bounded, then f is called a band-limited function.

2. General scaling functions

We give a characterization of supports of Fourier transforms of scaling functions as follows.

Theorem 2.1. *Let G be a bounded closed set in R^d. Then there is a scaling function φ with* $\operatorname{supp} \hat{\varphi} = G$ *if and only if*

$$\text{(i)}\quad G \subset 2G, \qquad \text{(ii)}\quad \bigcup_m 2^m G \simeq R^d, \qquad \text{(iii)}\quad G + 2\pi Z^d \simeq R^d \quad \textit{and}$$

$$\text{(vi)}\quad \left(G \Big\backslash \frac{1}{2}G\right) \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad \left(\nu \in Z^d\right)$$

hold.

In order to prove this theorem, we introduce some notations: for $m = 0, 1, \ldots,$

$$G_0^* = G \Big\backslash \frac{1}{2}G, \qquad G_m^* = 2^{-m}G_0^*, \qquad G_m = 2^{-m}G; \qquad E_m^* = G_m^* + 2\pi Z^d, \qquad E_m = G_m + 2\pi Z^d.$$

Since G is bounded, we can choose a $k \in Z^+$ such that $G \subset [-2^k\pi, 2^k\pi]^d$.

First we discuss the decomposition of G.

Lemma 2.2. *If G satisfies* (i) *and* (vi)*, then we have*

$$\text{(a)}\quad G = \left(\bigcup_{m=0}^{k-1} G_m^*\right) \cup G_k, \quad \textit{where } G_0^*, \ldots, G_{k-1}^*, G_k \textit{ are mutually disjoint.}$$

$$\text{(b)}\quad \frac{1}{2}G + 2\pi Z^d = \left(\bigcup_{m=1}^{k} E_m^*\right) \cup E_{k+1}, \qquad E_m^* \cap E_n^* \simeq \emptyset \quad (m \neq n,\ m, n \geqslant 0),$$

$$E_m^* \cap E_n \simeq \emptyset \quad (0 \leqslant m < n).$$

Proof. By (i), we have

$$G = \left(G \Big\backslash \frac{1}{2}G\right) \cup \frac{1}{2}G = G_0^* \cup G_1^* \cup \frac{1}{4}G = \cdots = \left(\bigcup_{m=0}^{k-1} G_m^*\right) \cup G_k$$

and clearly this is an union of $k + 1$ disjoint point sets, we get (a).

By (a), we have $\frac{1}{2}G = (\bigcup_{m=1}^{k} G_m^*) \cup G_{k+1}$. Furthermore,

$$\frac{1}{2}G + 2\pi Z^d = \left(\bigcup_{m=1}^{k} \left(G_m^* + 2\pi Z^d\right)\right) \cup \left(G_{k+1} + 2\pi Z^d\right).$$

For any m ($m \geqslant 0$), by (vi), we have

$$G_m^* \cap \bigl(G_{m+1} + 2\pi 2^{-m}\nu\bigr) \simeq \emptyset \quad \bigl(\nu \in Z^d\bigr).$$

Letting $\nu = 2^m\mu$ ($\mu \in Z^d$), we get $G_m^* \cap (G_{m+1} + 2\pi\mu) \simeq \emptyset$ ($\mu \in Z^d$). Again by $G_n \subset G_{m+1}$ $(m < n)$,

$$\bigl(G_m^* + 2\pi Z^d\bigr) \cap \bigl(G_n + 2\pi Z^d\bigr) \simeq \emptyset \quad (m < n). \tag{2.1}$$

From this and $E_n^* \subset E_n$, we get (b). □

Lemma 2.3. *Under the conditions of Lemma* 2.2, *we have*

$$\text{(a)} \quad (E_m^* \setminus G_m^*) \cap G \simeq \emptyset \quad (m \geqslant 0), \qquad \text{(b)} \quad G \cap (G_k + 2\pi\nu) \simeq \emptyset \quad (\nu \neq 0).$$

Hereafter $\nu \in Z^d \setminus \{0\}$ *is written simply as* $\nu \neq 0$.

Proof. For $m \geqslant 0$, by Lemma 2.2(b) and $G_n \subset E_n$, $G_n^* \subset E_n^*$, we conclude that

$$(E_m^* \setminus G_m^*) \cap G_n^* \simeq \emptyset \quad (n, m \geqslant 0), \qquad (E_m^* \setminus G_m^*) \cap G_n \simeq \emptyset \quad (0 \leqslant m < n).$$

So

$$(E_m^* \setminus G_m^*) \cap \left(\left(\bigcup_{n=0}^{l-1} G_n^*\right) \cup G_l\right) \simeq \emptyset \quad (0 \leqslant m < l).$$

Similar to the argument of Lemma 2.2(a), we have $(\bigcup_{n=0}^{l-1} G_n^*) \cup G_l = G$. So we get (a).

By $G \subset [-2^k\pi, 2^k\pi]^d$, we see that $G_k \subset [-\pi, \pi]^d$. So we obtain that $G_k \cap (G_k + 2\pi\nu) \simeq \emptyset$ ($\nu \neq 0$). On the other hand, by (2.1), we have $G_m^* \cap (G_k + 2\pi\nu) \simeq \emptyset$ ($m < k$, $\nu \in Z^d$). Hence we have

$$\left(\left(\bigcup_{m=0}^{k-1} G_m^*\right) \cup G_k\right) \cap (G_k + 2\pi\nu) \simeq \emptyset \quad (\nu \neq 0).$$

So we get (b). □

Next we discuss the decomposition of each G_m^* ($m \geqslant 0$).

Definition 2.4. For arbitrarily finitely many distinct points $\nu_0, \nu_1, \ldots, \nu_r$ ($0 \in \{\nu_l\}_0^r \subset Z^d$), define a point set $F_{\nu_0,\ldots,\nu_r}^m$ as follows

$$F_{\nu_0,\ldots,\nu_r}^m := \bigl\{\omega \in G_m^*\colon\ \omega + \pi\nu \in G_m^* \ \bigl(\nu \in \{\nu_l\}_0^r\bigr) \text{ and } \omega + \pi\nu \notin G_m^* \ \bigl(\nu \in Z^d \setminus \{\nu_l\}_0^r\bigr)\bigr\}.$$

Let $\omega \in G_m^*$. Since G is bounded and $G_m^* \subset G$, there only exist finitely many $\nu \in Z^d$ such that $\omega + \pi\nu \in G_m^*$, so ω lies in some $F_{\nu_0,\ldots,\nu_r}^m$. Finally, we have

$$G_m^* = \bigcup_{r=0}^{\infty} \bigcup_{0 \in \{\nu_j\}_0^r \subset Z^d} F_{\nu_0,\ldots,\nu_r}^m \tag{2.2}$$

and the right-hand side in (2.2) is a disjoint union.

Below we indicate that the union in (2.2) consists of finitely many nonempty point sets. Let

$$P^m := \bigl\{F_{\nu_0,\ldots,\nu_r}^m\colon\ 0 \in \{\nu_l\}_0^r \subset Z^d,\ r \geqslant 0 \text{ and } F_{\nu_0,\ldots,\nu_r}^m \neq \emptyset\bigr\}. \tag{2.3}$$

Note that $G \subset [-2^k\pi, 2^k\pi]^d$. If some ν_l in $\nu_0, \ldots, \nu_r$ ($0 \in \{\nu_l\}_0^r \subset Z^d$) satisfies $|\nu_l| > 2^{k+1}\sqrt{d}$, then for any $\omega \in G$, we have $\omega + \pi\nu_l \notin G$. Again by $G_m^* \subset G$, for any $\omega \in G_m^*$, we have $\omega + \pi\nu_l \notin G_m^*$. So, by Definition 2.4, $F_{\nu_0,\ldots,\nu_r}^m = \emptyset$. Finally, we see that P^m consists of finitely many point sets.

Lemma 2.5. *Let* $F_{\nu_0,\ldots,\nu_r}^m \in P^m$. *Then* $F_{\nu_0,\ldots,\nu_r}^m + \pi\nu_s \in P^m$ $(s = 0, \ldots, r)$ *and they are mutually disjoint.*

Proof. For each s, we see that $\omega \in F^m_{\nu_0,\dots,\nu_r} + \pi\nu_s$ is equivalent to $\omega - \pi\nu_s \in F^m_{\nu_0,\dots,\nu_r}$. Let

$$\alpha_{l,s} = \nu_l - \nu_s \quad (l = 0,\dots,r).$$

Then $\alpha_{s,s} = 0$. So $0 \in \{\alpha_{l,s}\}_0^r \subset Z^d$. By Definition 2.4, $\omega - \pi\nu_s \in F^m_{\nu_0,\dots,\nu_r}$ is equivalent to

$$\omega \in G_m^*, \quad \omega + \pi\nu \in G_m^* \quad \bigl(\nu \in \{\alpha_{l,s}\}_{l=0}^r\bigr) \quad \text{and} \quad \omega + \pi\nu \notin G_m^* \quad \bigl(\nu \in Z^d \setminus \{\alpha_{l,s}\}_{l=0}^r\bigr), \tag{2.4}$$

and (2.4) is equivalent to $\omega \in F^m_{\alpha_{0,s},\dots,\alpha_{r,s}}$. So

$$F^m_{\nu_0,\dots,\nu_r} + \pi\nu_s = F^m_{\alpha_{0,s},\dots,\alpha_{r,s}} \in P^m. \tag{2.5}$$

Let $s_1 \neq s_2$. Then $\nu_{s_1} \neq \nu_{s_2}$ and $\alpha_{l,s_1} \neq \alpha_{l,s_2}$ $(l = 0,\dots,r)$. So, by (2.5), we have

$$\bigl(F^m_{\nu_0,\dots,\nu_r} + \pi\nu_{s_1}\bigr) \cap \bigl(F^m_{\nu_0,\dots,\nu_r} + \pi\nu_{s_2}\bigr) = \emptyset. \qquad \square$$

Lemma 2.6. *There exist finitely many point sets* $A_j^m := F^m_{\nu_0^{(j)},\dots,\nu_{r_j}^{(j)}}$ $(j = 1,\dots,\lambda)$ *in* P^m *such that the decomposition* $G_m^* = \bigcup_{j=1}^{\lambda}\bigcup_{l=0}^{r_j}(A_j^m + \pi\nu_l^{(j)})$ *holds and this is a disjoint union.*

Proof. Take a point set $A_1^m = F^m_{\nu_0^{(1)},\dots,\nu_{r_1}^{(1)}} \in P^m$. By Lemma 2.5, the point sets $A_1^m + \pi\nu_l^{(1)} \in P^m$ $(l = 0,\dots,r_1)$ and they are mutually disjoint. Denote

$$S_1^m := \bigl\{A_1^m + \pi\nu_l^{(1)}\ (l = 0,\dots,r_1)\bigr\}. \tag{2.6}$$

Let $P_1^m := P^m \setminus S_1^m$. Take a point set $A_2^m = F^m_{\nu_0^{(2)},\dots,\nu_{r_2}^{(2)}} \in P_1^m$. We can prove that

$$A_2^m + \pi\nu_l^{(2)} \notin S_1^m \quad (l = 0,\dots,r_2). \tag{2.7}$$

If it is not true, then there exist l and n $(0 \leqslant l \leqslant r_2,\ 0 \leqslant n \leqslant r_1)$ such that

$$A_2^m + \pi\nu_l^{(2)} = A_1^m + \pi\nu_n^{(1)}. \tag{2.8}$$

By $A_2^m \subset G_m^*$, we see that

$$A_1^m + \pi\bigl(\nu_n^{(1)} - \nu_l^{(2)}\bigr) \subset G_m^*. \tag{2.9}$$

Since $A_1^m := F^m_{\nu_0^{(1)},\dots,\nu_{r_j}^{(1)}}$, by Definition 2.4, we know that there is s $(0 \leqslant s \leqslant r_1)$ such that $\nu_n^{(1)} - \nu_l^{(2)} = \nu_s^{(1)}$. Again by (2.8) and (2.6), we get

$$A_2^m = A_1^m + \pi\nu_s^{(1)} \in S_1^m.$$

This is contrary to $A_2^m \in P_1^m = P^m \setminus S_1^m$. So (2.7) holds. Denote $S_2^m = \{A_2^m + \pi\nu_l^{(2)}\ (l = 0,\dots,r_2)\}$.

Since P^m consists of finitely many point sets, repeating the above process, we finally can choose finitely many point sets $\{A_j^m\}_1^\lambda \subset P^m$ (λ is some natural number) such that

$$P^m = \bigcup_{j=1}^{\lambda} S_j^m, \quad \text{where } S_j^m = \bigl\{A_j^m + \pi\nu_l^{(j)}\ (l = 0,\dots,r_j)\bigr\}. \tag{2.10}$$

Namely, $P^m = \{A_j^m + \pi\nu_l^{(j)},\ l = 0,\dots,r_j;\ j = 1,\dots,\lambda\}$. By (2.2) and (2.3), we get the desired result. $\square$

We need also the following lemma.

Lemma 2.7. *Let* $\varphi \in L^2(R^d)$ *and a closed set* G *satisfy the following*:

$$\operatorname{supp}\hat\varphi = G, \tag{2.11}$$

$$\hat\varphi(2\omega) = H(\omega)\hat\varphi(\omega) \quad \text{a.e. } \omega \in R^d \ \bigl(H \in L^\infty\bigl(T^d\bigr)\bigr), \tag{2.12}$$

$$\sum_\nu \bigl|\hat\varphi(\omega + 2\pi\nu)\bigr|^2 = 1 \quad \text{a.e. } \omega \in R^d \tag{2.13}$$

and $\bigcup_m 2^m G \simeq R^d$. *Then* φ *is a scaling function.*

Proof. Let

$$V_m := \overline{\operatorname{span}}\{\varphi(2^m t - n),\ n \in Z^d\} \quad (m \in Z).$$

By (2.12), $V_m \subset V_{m+1}$. Again by (2.11) and $\bigcup_m 2^m G \simeq R^d$, we get

$$\overline{\bigcup_m} \bigcup_{f \in V_m} \operatorname{supp} \hat{f} = \overline{\bigcup_m} \operatorname{supp} \hat{\varphi}(2^{-m}\cdot) = \overline{\bigcup_m} 2^m G = R^d.$$

Applying a known result [4, p. 67], we obtain that $\overline{\bigcup_m} V_m = L^2(R^d)$. Again by (2.13), we conclude that φ is a scaling function. □

Proof of Theorem 2.1. *Sufficiency*. Suppose that (i)–(iii) and (vi) hold.

Since G is bounded, we can choose a $k \in Z^+$ such that $G \subset [-2^k\pi, 2^k\pi]^d$. By Lemma 2.7, it is enough that we directly construct $\varphi \in L^2(R^d)$ and $H \in L^\infty(T^d)$ satisfying (2.11)–(2.13). This process of construction is split into the following five steps:

Step 1. Define $\hat{\varphi}(\omega) = 0$ $(\omega \notin G)$.

Step 2. Define $\hat{\varphi}(\omega) = 1$ $(\omega \in G_k)$. For $\omega \in G_k$, by Lemma 2.3(b), we see that $\omega + 2\pi\nu \notin G$ $(\nu \neq 0)$. So

$$\hat{\varphi}(\omega) = \begin{cases} 0, & \omega \in G_k + 2\pi\nu\ (\nu \neq 0), \\ 1, & \omega \in G_k. \end{cases} \tag{2.14}$$

Hence $\hat{\varphi}(\omega)$ has been defined on E_k and $\sum_\nu |\hat{\varphi}(\omega + 2\pi\nu)|^2 = |\hat{\varphi}(\omega)|^2 = 1$ $(\omega \in G_k)$, i.e., (2.13) holds for $\omega \in G_k$.

Define $H(\omega) = 1$ $(\omega \in E_{k+1})$. If $\omega \in G_{k+1}$, then $2\omega \in G_k$. By $G_{k+1} \subset G_k$ and (2.14), $\hat{\varphi}(2\omega) = \hat{\varphi}(\omega) = 1$. Hence (2.12) holds for $\omega \in G_{k+1}$. If $\omega \in G_{k+1} + 2\pi\nu$ $(\nu \neq 0)$, then $2\omega \in G_k + 4\pi\nu$ $(\nu \neq 0)$, so, by (2.14), $\hat{\varphi}(2\omega) = \hat{\varphi}(\omega) = 0$, clearly, (2.12) holds for $\omega \in G_{k+1} + 2\pi\nu(\nu \neq 0)$. Therefore, (2.12) holds for $\omega \in E_{k+1}$.

Step 3. Let $P(n)$ be the following proposition:

Proposition. *For each n $(0 \leqslant n \leqslant k)$, we can define a function $\hat{\varphi}(\omega)$ on G_n^* and a function $H(\omega)$ with period 2π on 2π-periodic point set E_{n+1}^* such that* (2.13) *holds on G_n^* and* (2.12) *holds on E_{n+1}^*, and*

$$0 < C_n \leqslant \hat{\varphi}(\omega) \leqslant D_n < \infty \quad \textit{for } \omega \in G_n^*; \qquad 0 < \tilde{C}_n \leqslant H(\omega) \leqslant \tilde{D}_n < \infty \quad \textit{for } \omega \in E_{n+1}^*,$$

where C_n, D_n, $\tilde{C}_n$ and $\tilde{D}_n$ are constants.

By Lemma 2.3(a) and $\hat{\varphi}(\omega) = 0$ $(\omega \notin G)$, we know that for $\omega \in E_n^* \setminus G_n^*$, $\hat{\varphi}(\omega) = 0$. Hence if we define $\hat{\varphi}(\omega)$ on G_n^*, then we have defined $\hat{\varphi}(\omega)$ on E_n^*.

Since $G_k^* \subset G_k$ and $E_{k+1}^* \subset E_{k+1}$, by Step 2, we see that Proposition $P(k)$ is true and

$$C_k = D_k = \tilde{C}_k = \tilde{D}_k = 1.$$

Now suppose that $0 < m \leqslant k$ and that $P(m)$ is true, we will prove that $P(m-1)$ is true.

First we define $H(\omega)$ on G_m^*. By Lemma 2.6, we only need to define $H(\omega)$ on each $A_j^m + \pi\nu_l^{(j)}$.

Since $P(m)$ is true, $\hat{\varphi}(\omega)$ has been defined on G_m^* and $C_m \leqslant \hat{\varphi}(\omega) \leqslant D_m$. Again by Lemma 2.6, we know that for $\omega \in A_j^m$, $l = 0, \ldots, r_j$, the values of $\hat{\varphi}(\omega + \pi\nu_l^{(j)})$ have been defined and $C_m \leqslant \hat{\varphi}(\omega + \pi\nu_l^{(j)}) \leqslant D_m$. Write

$$\nu_l^{(j)} = 2u_l^{(j)} + \alpha_l^{(j)}, \quad \text{where } u_l^{(j)} \in Z^d \text{ and } \alpha_l^{(j)} \in \{0, 1\}^d. \tag{2.15}$$

We can define the functions $H(\omega + \pi\alpha_l^{(j)})$ $(\omega \in A_j^m)$ such that for $\omega \in A_j^m$,

$$\sum_{l=0}^{r_j} b_l^{(j)}(\omega) \big| H(\omega + \pi\alpha_l^{(j)}) \big|^2 = 1, \quad \text{where } b_l^{(j)}(\omega) = \big| \hat{\varphi}(\omega + \pi\nu_l^{(j)}) \big|^2 \tag{2.16}$$

and

$$0 < \tilde{C}_{m-1}^{(j)} \leqslant H(\omega + \pi\alpha_l^{(j)}) \leqslant \tilde{D}_{m-1}^{(j)} < \infty \quad (l = 0, \ldots, r_j), \quad \text{where } \tilde{C}_{m-1}^{(j)},\ \tilde{D}_{m-1}^{(j)} \text{ are constants.} \tag{2.17}$$

In Remark 2.8, we will state the construction of the functions $H(\omega+\pi\alpha_l^{(j)})$ in detail.

Again define

$$H\big(\omega+\pi\nu_l^{(j)}\big)=H\big(\omega+\pi\alpha_l^{(j)}\big)\quad\big(\omega\in A_j^m\big),\ l=0,\dots,r_j. \tag{2.18}$$

By Lemma 2.6, $H(\omega)$ has been defined on G_m^* and

$$\tilde C_{m-1}\leqslant H(\omega)\leqslant \tilde D_{m-1}\quad(\omega\in G_m^*),\quad\text{where } \tilde C_{m-1}=\min_{1\leqslant j\leqslant\lambda}\big\{\tilde C_{m-1}^{(j)}\big\},\ \tilde D_{m-1}=\max_{1\leqslant j\leqslant\lambda}\big\{\tilde D_{m-1}^{(j)}\big\},$$

where λ is stated in Lemma 2.6.

Below we will prove that if $\omega\in G_m^*$ and $\omega+2\pi\tilde\nu\in G_m^*$ for some $\tilde\nu\in Z^d$, then $H(\omega+2\pi\tilde\nu)=H(\omega)$.

By $\omega\in G_m^*$ and Lemma 2.6, we know that there exist j and n such that

$$\omega=\omega_0+\pi\nu_n^{(j)},\quad\text{where } \omega_0\in A_j^m. \tag{2.19}$$

Since $\omega+2\pi\tilde\nu=\omega_0+\pi(2\tilde\nu+\nu_n^{(j)})$ and $\omega+2\pi\tilde\nu\in G_m^*$, we have $\omega_0+\pi(2\tilde\nu+\nu_n^{(j)})\in G_m^*$. Again since $\omega_0\in A_j^m$, by Definition 2.4, we know that there is some s $(0\leqslant s\leqslant r_j)$ such that

$$2\tilde\nu+\nu_n^{(j)}=\nu_s^{(j)}. \tag{2.20}$$

Again by (2.18)–(2.20), we get

$$H(\omega)=H\big(\omega_0+\pi\nu_n^{(j)}\big)=H\big(\omega_0+\pi\alpha_n^{(j)}\big),$$
$$H(\omega+2\pi\tilde\nu)=H\big(\omega_0+\pi\big(2\tilde\nu+\nu_n^{(j)}\big)\big)=H\big(\omega_0+\pi\nu_s^{(j)}\big)=H\big(\omega_0+\pi\alpha_s^{(j)}\big).$$

But by (2.15) and (2.20),

$$\nu_s^{(j)}=2\big(\tilde\nu+u_n^{(j)}\big)+\alpha_n^{(j)}\quad\big(\tilde\nu+u_n^{(j)}\in Z^d,\ \alpha_n^{(j)}\in\{0,1\}^d\big),$$
$$\nu_s^{(j)}=2u_s^{(j)}+\alpha_s^{(j)}\quad\big(u_s^{(j)}\in Z^d,\ \alpha_s^{(j)}\in\{0,1\}^d\big).$$

By uniqueness of decomposition in (2.15), we have $\alpha_n^{(j)}=\alpha_s^{(j)}$. From this, we have $H(\omega)=H(\omega+2\pi\tilde\nu)$.

Define $H(\omega+2\pi\nu)=H(\omega)$, $\omega\in G_m^*$ $(\nu\in Z^d)$. So $H(\omega)$ is well defined on E_m^* and $H(\omega)$ is a function with 2π-period on 2π-periodic point set E_m^* and $\tilde C_{m-1}\leqslant H(\omega)\leqslant\tilde D_{m-1}$ $(\omega\in E_m^*)$.

Now we define $\hat\varphi(\omega)$ on G_{m-1}^*. Since $P(m)$ is true, $C_m\leqslant\hat\varphi(\omega)\leqslant D_m$ on G_m^*. Let

$$\hat\varphi(2\omega)=H(\omega)\hat\varphi(\omega)\quad(\omega\in G_m^*). \tag{2.21}$$

Then the function $\hat\varphi(\omega)$ is defined on G_{m-1}^* $(G_{m-1}^*=2G_m^*)$ and

$$C_{m-1}\leqslant\hat\varphi(\omega)\leqslant D_{m-1}\quad\text{on } G_{m-1}^*,\quad\text{where } C_{m-1}=C_m\tilde C_{m-1},\ D_{m-1}=D_m\tilde D_{m-1}.$$

By Step 1 and Lemma 2.3(a), we get $\hat\varphi(\omega)=0$ $(\omega\in E_{m-1}^*\setminus G_{m-1}^*)$. So the function $\hat\varphi(\omega)$ has defined on E_{m-1}^*.

We prove that (2.12) holds on E_m^*. Let $\omega\in E_m^*$. If $\omega\in G_m^*$, then by (2.21), we know that (2.12) holds. If $\omega\notin G_m^*$, then $\omega\in E_m^*\setminus G_m^*$, by Lemma 2.3(a), $\omega\notin G$. By $G\subset 2G$, we have $2\omega\notin G$, so $\hat\varphi(2\omega)=\hat\varphi(\omega)=0$, (2.12) also holds.

Finally we prove that (2.13) holds on G_{m-1}^*.

Let $\omega\in A_j^m=F^m_{\nu_0^{(j)},\dots,\nu_{r_j}^{(j)}}$. By Definition 2.4, we have

$$\omega\in G_m^*,\qquad \omega+\pi\nu\in G_m^*\quad\big(\nu\in\big\{\nu_l^{(j)}\big\}_0^{r_j}\big),\qquad \omega+\pi\nu\notin G_m^*\quad\big(\nu\notin\big\{\nu_l^{(j)}\big\}_0^{r_j}\big). \tag{2.22}$$

Again by $2G_m^*=G_{m-1}^*$, we obtain that $2\omega\in G_{m-1}^*$, so

$$2\omega+2\pi\nu\in E_{m-1}^*\quad\big(\nu\in Z^d\big).$$

On the other hand, by the third formula in (2.22), we get

$$2\omega+2\pi\nu\notin G_{m-1}^*\quad\big(\nu\notin\big\{\nu_l^{(j)}\big\}_0^{r_j}\big).$$

So $2\omega+2\pi\nu\in E^*_{m-1}\setminus G^*_{m-1}$ ($\nu\notin\{\nu_l\}_0^{r_j}$). By Lemma 2.3(a), we see that $2\omega+2\pi\nu\notin G$ ($\nu\notin\{\nu_l\}_0^{r_j}$), so $\hat\varphi(2\omega+2\pi\nu)=0$ ($\nu\notin\{\nu_l^{(j)}\}_0^{r_j}$). Furthermore

$$\sum_\nu\left|\hat\varphi(2\omega+2\pi\nu)\right|^2=\sum_{l=0}^{r_j}\left|\hat\varphi\left(2\omega+2\pi\nu_l^{(j)}\right)\right|^2. \tag{2.23}$$

Noticing that the second formula in (2.22), we conclude by (2.21) and (2.18) that

$$\hat\varphi\left(2\omega+2\pi\nu_l^{(j)}\right)=H\left(\omega+\pi\nu_l^{(j)}\right)\hat\varphi\left(\omega+\pi\nu_l^{(j)}\right)=H\left(\omega+\pi\alpha_l^{(j)}\right)\hat\varphi\left(\omega+\pi\nu_l^{(j)}\right).$$

From this and (2.23), (2.16), we obtain that for $\omega\in A_j^m$,

$$\sum_\nu\left|\hat\varphi(2\omega+2\pi\nu)\right|^2=1. \tag{2.24}$$

Again by the periodicity of the sum $\sum_\nu|\hat\varphi(2\omega+2\pi\nu)|^2$, (2.24) also holds for $\omega\in A_j^m+\pi\nu_l^{(j)}$. So, by Lemma 2.6, (2.24) holds for $\omega\in G^*_m$. Noticing that $2G^*_m=G^*_{m-1}$, we know that (2.13) holds on G^*_{m-1}. So $P(m-1)$ is true.

We have shown that $P(m)$ implies $P(m-1)$. Hence Proposition $P(n)$ is true for $0\leqslant n\leqslant k$.

Step 4. By Proposition $P(n)$ ($0\leqslant n\leqslant k$) and Steps 1 and 2, using Lemma 2.2, we see that $\hat\varphi(\omega)$ has been defined on R^d and the function $H(\omega)$ with period 2π has been defined on 2π-periodic point set $\frac12G+2\pi Z^d$ and (2.13) holds on G and (2.12) holds on $\frac12G+2\pi Z^d$.

By Proposition $P(n)$ and Step 2, we have $C_n\leqslant\hat\varphi(\omega)\leqslant D_n$ on G^*_n ($n=0,\dots,k-1$) and $\hat\varphi(\omega)=1$ on G_k. By Lemma 2.2(a), we see that $\hat\varphi(\omega)$ is bounded on G and $\hat\varphi(\omega)>0$ on G. Again since G is a bounded closed set and $\hat\varphi(\omega)=0$ ($\omega\notin G$), we know that $\hat\varphi(\omega)\in L^2(R^d)$ and $\operatorname{supp}\hat\varphi(\omega)=G$, i.e., (2.11) holds. Similarly, we know that $H(\omega)$ is bounded on $\frac12G+2\pi Z^d$.

Step 5. Define $H(\omega)=0$ ($\omega\in G^*_0+2\pi Z^d$). By (iii) and (vi), we have

$$\left(G^*_0+2\pi Z^d\right)\cup\left(\frac12G+2\pi Z^d\right)=G+2\pi Z^d\simeq R^d,\qquad\left(G^*_0+2\pi Z^d\right)\cap\left(\frac12G+2\pi Z^d\right)\simeq\emptyset. \tag{2.25}$$

From this, we know that H has been well defined on R^d and $H\in L^\infty(T^d)$.

Below we prove that both (2.12) and (2.13) hold on R^d.

From Step 4, we have known that (2.12) holds on $\frac12G+2\pi Z^d$. For $\omega\in G^*_0+2\pi Z^d$, we have $2\omega\in2G^*_0+4\pi Z^d$. By the second formula in (2.25), we have $(2G^*_0+4\pi Z^d)\cap G\simeq\emptyset$. Hence $\hat\varphi(2\omega)=0$. Again by $H(\omega)=0$ ($\omega\in G^*_0+2\pi Z^d$), we know that (2.12) holds also on $G^*_0+2\pi Z^d$. Therefore by the first formula in (2.25), we conclude that (2.12) holds for $\omega\in R^d$.

From Step 4, we have known that (2.13) holds on G. Since the sum $\sum_\nu|\hat\varphi(\omega+2\pi\nu)|^2$ is a 2π-periodic function, we easily conclude by (iii) that (2.13) holds on R^d.

Up to now, we have constructed $\varphi\in L^2(R^d)$ and $H\in L^\infty(T^d)$ satisfying (2.11)–(2.13). Sufficiency is proved.

Necessity. Suppose that φ is a scaling function with $\operatorname{supp}\hat\varphi=G$. We have known that (i)–(iii) hold (see Remark 1.4). For a.e. $\omega\in G\setminus\frac12G$, we see that $2\omega\in2G\setminus G$, so $\hat\varphi(2\omega)=0$, $\hat\varphi(\omega)\neq0$. Again by the refined equation $\hat\varphi(2\omega)=H(\omega)\hat\varphi(\omega)$ ($H\in L^2(T^d)$), we get

$$H(\omega)=0\quad\text{for a.e. }\omega\in G\setminus\frac12G. \tag{2.26}$$

For a.e. $\omega\in\frac12G$, by $2\omega\in G$ and (i), we have $\hat\varphi(2\omega)\neq0$, $\hat\varphi(\omega)\neq0$. Again by $\hat\varphi(2\omega)=H(\omega)\hat\varphi(\omega)$, we know that $H(\omega)\neq0$ a.e. $\omega\in\frac12G$. Since $H(\omega)$ is a periodic function with period 2π,

$$H(\omega)\neq0\quad\text{for a.e. }\omega\in\frac12G+2\pi\nu\ \left(\nu\in Z^d\right). \tag{2.27}$$

Combining (2.26) with (2.27), we have $(G\setminus\frac12G)\cap(\frac12G+2\pi\nu)\simeq\emptyset$ ($\nu\in Z^d$), i.e., (vi) holds. Necessity is proved. □

Remark 2.8. Here we construct the functions $H(\omega+\pi\alpha_l^{(j)})$ ($\omega\in A_j^m$) such that (2.16) and (2.17) hold.

Let

$$N_j(\beta) = \left\{ l \in \{0, \ldots, r_j\};\ \alpha_l^{(j)} = \beta \right\} \quad \left(\beta \in \{0,1\}^d\right).$$

Denote the set of β satisfying $N_j(\beta) \neq \emptyset$ by $\{\beta_\tau\}_1^{h_j}$. It is clear that

$$\{0, \ldots, r_j\} = \bigcup_{\tau=1}^{h_j} N_j(\beta_\tau), \qquad \alpha_l^{(j)} = \beta_\tau \quad \left(l \in N_j(\beta_\tau)\right).$$

Let $Q_{j,\tau}(\omega) = \sum_{l \in N_j(\beta_\tau)} b_l^{(j)}(\omega)$, where $b_l^{(j)}(\omega) = |\hat{\varphi}(\omega + \pi \nu_l^{(j)})|^2$. In view of

$$C_m \leqslant \hat{\varphi}\left(\omega + \pi \nu_l^{(j)}\right) \leqslant D_m \quad \left(\omega \in A_j^m\right), \qquad N_j(\beta) \subset \{0, \ldots, r_j\},$$

we obtain that

$$C_m^2 \leqslant Q_{j,\tau}(\omega) \leqslant (r_j + 1) D_m^2 \quad \left(\omega \in A_j^m\right).$$

For $\omega \in A_j^m$, define

$$H(\omega + \pi \beta_\tau) = \frac{1}{\sqrt{h_j Q_{j,\tau}(\omega)}} \quad (\tau = 1, \ldots, h_j).$$

By $\alpha_l^{(j)} = \beta_\tau$ $(l \in N_j(\beta_\tau))$, we have

$$H\left(\omega + \pi \alpha_l^{(j)}\right) = H(\omega + \pi \beta_\tau) = \frac{1}{\sqrt{h_j Q_{j,\tau}(\omega)}} \quad \left(\omega \in A_j^m\right),\ l \in N_j(\beta_\tau).$$

In view of $\{0, \ldots, r_j\} = \bigcup_{\tau=1}^{h_j} N_j(\beta_\tau)$, for $\omega \in A_j^m$, we have

$$\frac{1}{\sqrt{(r_j + 1) h_j} D_m} \leqslant H\left(\omega + \pi \alpha_l^{(j)}\right) \leqslant \frac{1}{C_m}$$

and

$$\sum_{l=0}^{r_j} b_l^{(j)} \left| H\left(\omega + \pi \alpha_l^{(j)}\right)\right|^2 = \sum_{\tau=1}^{h_j} \left(\sum_{l \in N_j(\beta_\tau)} b_l^{(j)}(\omega) \right) \left| H(\omega + \pi \beta_\tau) \right|^2 = \sum_{\tau=1}^{h_j} Q_{j,\tau}(\omega) \left| H(\omega + \pi \beta_\tau) \right|^2 = 1.$$

Namely, (2.16) and (2.17) are valid.

Remark 2.9. In the proof of sufficiency in Theorem 2.1, we present a general "approach" to the construction of the scaling function φ with $\operatorname{supp} \hat{\varphi} = G$.

There is no function available which is a band-limited scaling function but is not a W-type scaling function, below we present such an example.

Example 2.10. Let $G \subset R^2$ and

$$G = \tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3, \quad \text{where } \tilde{E}_1 = [-\pi, \pi]^2,\ \tilde{E}_2 = \left[\frac{3}{2}\pi, \frac{7}{4}\pi\right] \times \left[-\frac{1}{8}\pi, \frac{1}{8}\pi\right],\ \tilde{E}_3 = 2\tilde{E}_2.$$

It is easy to check directly that G satisfies the following formulas:

$$G \subset 2G, \quad \bigcup_m 2^m G = R^2, \quad G + 2\pi Z^2 = R^2 \quad \text{and} \quad \left(G \Big\backslash \frac{1}{2} G\right) \cap \left(\frac{1}{2} G + 2\pi Z^2\right) \simeq \emptyset.$$

By Theorem 2.1, there is a scaling function $\varphi \in L^2(R^2)$ with $\operatorname{supp} \hat{\varphi} = G$. However, since $\frac{1}{2}G \cap (\frac{1}{2}G + 2\pi e_1) = \tilde{E}_2 \not\simeq \emptyset$ $(e_1 = (1, 0))$, by Walter's theorem, φ is not a W-type scaling function.

Now using the method presented by us in the proof of Theorem 2.1, we directly construct such scaling function φ. By $G \subset [-4\pi, 4\pi]^2$, we can choose $k = 2$. We decompose G into three disjoint point sets:

$$G = G_0^* \cup G_1^* \cup G_2,$$

where $G_0^* = G \setminus \frac{1}{2}G = ((\tilde{E}_1 \setminus \frac{1}{2}\tilde{E}_1) \setminus \frac{1}{2}\tilde{E}_2) \cup 2\tilde{E}_2$ and $G_1^* = \frac{1}{2}G_0^*$ and $G_2 = \frac{1}{4}G$.

Let $G_2^* = \frac{1}{4}G_0^*$. We again decompose G_1^* and G_2^*. By Definition 2.4, it is easy to check that

$$\begin{aligned} &F^1_{0,-2e_1} = \tilde{E}_2, \qquad F^1_{0,2e_1} = \tilde{E}_2 - 2\pi e_1, \qquad F^1_0 = G_1^* \setminus B, \\ &F^2_{0,-e_1} = \frac{1}{2}\tilde{E}_2, \qquad F^2_{0,e_1} = \frac{1}{2}\tilde{E}_2 - \pi e_1, \qquad F^2_0 = G_2^* \setminus \frac{1}{2}B, \end{aligned} \tag{2.28}$$

where $B = \tilde{E}_2 \cup (\tilde{E}_2 - 2\pi e_1)$ and $B \subset G_1^*$. From this, we get

$$G_1^* = F^1_{0,-2e_1} \cup \left(F^1_{0,-2e_1} - 2\pi e_1\right) \cup F^1_0, \qquad G_2^* = F^2_{0,-e_1} \cup \left(F^2_{0,-e_1} - \pi e_1\right) \cup F^2_0. \tag{2.29}$$

Let

$$\hat{\varphi}(\omega) = 0 \quad (\omega \notin G), \qquad \hat{\varphi}(\omega) = 1 \quad (\omega \in G_2). \tag{2.30}$$

Now we define $\hat{\varphi}(\omega)$ on G_1^*. According to the method given in Theorem 2.1, based on the decomposition of G_2^* in (2.29), we first define $H(\omega)$ on G_2^* as follows.

Let $\omega \in F^2_{0,-e_1}$. By (2.29) and $G_2^* \subset G_2$, we see that $\omega \in G_2$ and $\omega - \pi e_1 \in G_2$. From this and (2.30), we have $\hat{\varphi}(\omega) = \hat{\varphi}(\omega - \pi e_1) = 1$. We choose $H(\omega) = H(\omega - \pi e_1) = \frac{1}{\sqrt{2}}$ $(\omega \in F^2_{0,-e_1})$, so

$$\left|\hat{\varphi}(\omega)\right|^2\left|H(\omega)\right|^2 + \left|\hat{\varphi}(\omega - \pi e_1)\right|^2\left|H(\omega - \pi e_1)\right|^2 = 1 \quad \left(\omega \in F^2_{0,-e_1}\right).$$

Let $\omega \in F^2_0$. Since $F^2_0 \subset G_2^* \subset G_2$, by (2.30), $\hat{\varphi}(\omega) = 1$. We choose $H(\omega) = 1$ $(\omega \in F^2_0)$, so

$$\left|\hat{\varphi}(\omega)\right|^2\left|H(\omega)\right|^2 = 1 \quad \left(\omega \in F^2_0\right).$$

By (2.28), we see that $F^2_{0,-e_1} \cup (F^2_{0,-e_1} - \pi e_1) = \frac{1}{2}B$ and $F^2_0 = G_2^* \setminus \frac{1}{2}B$. Hence we have

$$H(\omega) = \begin{cases} 1, & \omega \in G_2^* \setminus \frac{1}{2}B, \\ \frac{1}{\sqrt{2}}, & \omega \in \frac{1}{2}B. \end{cases} \tag{2.31}$$

Let $\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega)$ $(\omega \in G_2^*)$.

By (2.30) and $G_2^* \subset G_2$, we have $\hat{\varphi}(\omega) = 1$ $(\omega \in G_2^*)$. So $\hat{\varphi}(2\omega) = H(\omega)$ $(\omega \in G_2^*)$. Again by (2.31) and $G_1^* = 2G_2^*$, we get

$$\hat{\varphi}(\omega) = \begin{cases} 1, & \omega \in G_1^* \setminus B, \\ \frac{1}{\sqrt{2}}, & \omega \in B. \end{cases} \tag{2.32}$$

Similar to the above process, based on the decomposition of G_1^* in (2.29) and $G_0^* = 2G_1^*$, we can define $\hat{\varphi}(\omega)$ on G_0^* as follows

$$\hat{\varphi}(\omega) = \begin{cases} 1, & \omega \in G_0^* \setminus 2B, \\ \frac{1}{\sqrt{2}}, & \omega \in 2B. \end{cases} \tag{2.33}$$

Since $(G_0^* \setminus 2B) \cup (G_1^* \setminus B) \cup G_2 = G \setminus (B \cup 2B)$, combining (2.30), (2.32) with (2.33), we have constructed the scaling function φ with $\operatorname{supp}\hat{\varphi} = G$ as follows

$$\hat{\varphi}(\omega) = \begin{cases} 1, & \omega \in G \setminus (B \cup 2B), \\ \frac{1}{\sqrt{2}}, & \omega \in B \cup 2B, \\ 0, & \omega \notin G, \end{cases}$$

where $B = \tilde{E}_2 \cup (\tilde{E}_2 - 2\pi e_1)$, $e_1 = (1, 0)$.

Below we discuss further the necessary conditions for G to be the support of the Fourier transform of the scaling function. These necessary conditions will be used in Sections 3 and 4.

Theorem 2.11. *Let φ be a band-limited scaling function with* $\operatorname{supp}\hat{\varphi} = G$. *Then*

$$\text{(a)}\quad 0 \in G, \qquad \text{(b)}\quad G + 2\pi Z^d = R^d, \qquad \text{(c)}\quad \left(G^o \Big\backslash \frac{1}{2}G\right) \cap \left(\frac{1}{2}G^o + 2\pi Z^d\right) = \emptyset \quad \textit{if } |\partial G| = 0,$$

where G^o is the interior of G and ∂G is the boundary of G.

Proof. Since φ be a scaling function with $\operatorname{supp}\hat{\varphi} = G$, by Theorem 2.1, (i)–(iii) and (vi) hold.

If (a) is not true, then $0 \notin G$. Since G is a closed set, there must be a point $\omega_0 \in G$ such that $\rho(0, \omega_0) = \rho(0, G)$ and $\omega_0 \neq 0$, where ρ is the distance. Hence $\frac{\omega_0}{2} \notin G$. On the other hand, by (i) and $\omega_0 \in G$ we see that $\frac{\omega_0}{2} \in \frac{1}{2}G \subset G$. This is a contradiction. So we get (a).

If a point $p \notin G + 2\pi Z^d$, then by (iii), we see that p is a limit point of $G + 2\pi Z^d$. Since G is bounded, for any ball $\bar{D}(p, \delta)$ with center p, there exists an integer $k > 0$ such that $(G + 2\pi\nu) \cap \bar{D}(p, \delta) = \emptyset$ ($\|\nu\| > k$), where $\|\nu\| = \sqrt{\nu_1^2 + \cdots + \nu_d^2}$ ($\nu = (\nu_1, \ldots, \nu_d)$). So p is a limit point of $H_k := \bigcup_{\|\nu\| \leqslant k}(G + 2\pi\nu)$. Since H_k is an union of finitely many closed sets, H_k is a closed set, further $p \in H_k$. So there exists ν_0 ($\|\nu_0\| \leqslant k$) such that $p \in G + 2\pi\nu_0$. This is contrary to $p \notin G + 2\pi Z^d$. So we get (b).

By (vi) and $|\partial G| = 0$, we have

$$\left(G^o \Big\backslash \frac{1}{2}G\right) \cap \left(\frac{1}{2}G^o + 2\pi\nu\right) \simeq \emptyset \quad \left(\nu \in Z^d\right),$$

also, $(G^o \setminus \frac{1}{2}G) \cap (\frac{1}{2}G^o + 2\pi\nu) = \emptyset$ ($\nu \in Z^d$) since $(G^o \setminus \frac{1}{2}G) \cap (\frac{1}{2}G^o + 2\pi\nu)$ is an open set. So we get (c). □

3. W-type scaling functions and convex supports

3.1. W-type scaling functions

We give a characterization of the supports of Fourier transforms of W-type scaling functions as follows.

Theorem 3.1. *Let G be a bounded closed set in R^d. Then there exists a W-type scaling function φ with* $\operatorname{supp}\hat{\varphi} = G$ *if and only if*

$$\text{(i)}\quad G \subset 2G, \qquad \text{(ii)}\quad \bigcup_m 2^m G \simeq R^d, \qquad \text{(iii)}\quad G + 2\pi Z^d \simeq R^d \quad \textit{and}$$

$$\text{(vii)}\quad G \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad \left(\nu \in Z^d \setminus \{0\}\right)$$

hold.

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let G be a bounded closed set and φ be a scaling function with* $\operatorname{supp}\hat{\varphi} = G$. *If* (vii) *holds, then φ is a W-type scaling function.*

Proof. For a.e. $\omega \in G$, by (vii), we get $\frac{\omega}{2} + 2\pi\nu \notin G$ ($\nu \neq 0$). From this and $\operatorname{supp}\hat{\varphi} = G$, it follows that

$$\hat{\varphi}\left(\frac{\omega}{2} + 2\pi\nu\right) = 0, \quad \text{a.e. } \omega \in G\ (\nu \neq 0).$$

Again since φ is a scaling function, we have

$$\left|\hat{\varphi}\left(\frac{\omega}{2}\right)\right|^2 + \sum_{\nu \neq 0}\left|\hat{\varphi}\left(\frac{\omega}{2} + 2\pi\nu\right)\right|^2 = \sum_{\nu}\left|\hat{\varphi}\left(\frac{\omega}{2} + 2\pi\nu\right)\right|^2 = 1 \quad \left(\text{a.e. } \omega \in R^d\right).$$

This implies that

$$\left|\hat{\varphi}\left(\frac{\omega}{2}\right)\right| = 1 \quad \text{a.e. } \omega \in G. \tag{3.1}$$

Denote

$$g(\omega) := \operatorname{Arg}\hat{\varphi}(\omega). \tag{3.2}$$

For any $\alpha \in R^d$, let

$$m_\alpha\left(\frac{\omega}{2}\right) := \sum_\nu \hat{\varphi}(\omega + 4\pi\nu)e^{-i\alpha\cdot(\omega+4\pi\nu)}e^{-ig(\frac{\omega}{2}+2\pi\nu)} \quad \left(\omega \in R^d\right).$$

Since φ is band-limited, the above sum has only finitely many nonzero terms and so $m_\alpha \in L^2(T^d)$. From the above formula, we get

$$\hat{\varphi}\left(\frac{\omega}{2}\right)m_\alpha\left(\frac{\omega}{2}\right) = \sum_\nu \hat{\varphi}(\omega + 4\pi\nu)\hat{\varphi}\left(\frac{\omega}{2}\right)e^{-i\alpha\cdot(\omega+4\pi\nu)}e^{-ig(\frac{\omega}{2}+2\pi\nu)} \quad \left(\omega \in R^d\right). \tag{3.3}$$

By (vii), we get $2G \cap (G + 4\pi\nu) \simeq \emptyset$ $(\nu \neq 0)$ which is equivalent to $(2G + 4\pi\nu) \cap G \simeq \emptyset$ $(\nu \neq 0)$. So for a.e. $\omega \in 2G$, we see that $\omega + 4\pi\nu \notin G$ $(\nu \neq 0)$. By $\operatorname{supp}\hat{\varphi} = G$, we have $\hat{\varphi}(\omega + 4\pi\nu) = 0$ (a.e. $\omega \in 2G$). For a.e. $\omega \notin 2G$, we have $\hat{\varphi}(\frac{\omega}{2}) = 0$. Therefore,

$$\hat{\varphi}(\omega + 4\pi\nu)\hat{\varphi}\left(\frac{\omega}{2}\right) = 0 \quad \text{a.e. } \omega \in R^d \ (\nu \neq 0).$$

From this and (3.2) and (3.3), it follows that

$$\hat{\varphi}\left(\frac{\omega}{2}\right)m_\alpha\left(\frac{\omega}{2}\right) = \hat{\varphi}(\omega)\hat{\varphi}\left(\frac{\omega}{2}\right)e^{-i\alpha\cdot\omega}e^{-ig(\frac{\omega}{2})} = \hat{\varphi}(\omega)\left|\hat{\varphi}\left(\frac{\omega}{2}\right)\right|e^{-i\alpha\cdot\omega} \quad \text{a.e. } \omega \in R^d. \tag{3.4}$$

Combining this with (3.1),

$$\hat{\varphi}\left(\frac{\omega}{2}\right)m_\alpha\left(\frac{\omega}{2}\right) = \hat{\varphi}(\omega)e^{-i\alpha\cdot\omega} \quad \text{for a.e. } \omega \in G.$$

Noticing that $\hat{\varphi}(\omega) = 0$ for a.e. $\omega \notin G$, we obtain by (3.4) that $\hat{\varphi}(\frac{\omega}{2})m_\alpha(\frac{\omega}{2}) = 0 = \hat{\varphi}(\omega)e^{-i\alpha\cdot\omega}$ for a.e. $\omega \notin G$. So

$$\hat{\varphi}\left(\frac{\omega}{2}\right)m_\alpha\left(\frac{\omega}{2}\right) = \hat{\varphi}(\omega)e^{-i\alpha\cdot\omega} \quad \text{for a.e. } \omega \in R^d.$$

Namely, $\hat{\varphi}(\frac{\omega}{2})m_\alpha(\frac{\omega}{2}) = (\varphi(\cdot - \alpha))^\wedge(\omega)$, so $\varphi(t - \alpha) \in V_1$. Since α is arbitrary, from this we can conclude that for any $f \in V_0$ and $\alpha \in R^d$, we have $f(t - \alpha) \in V_1$. By Definition 1.3, φ is a W-type scaling function. □

Proof of Theorem 3.1. If φ is a W-type scaling function, then by Walter's theorem, (i)–(iii) and (v) hold. Again since a W-type scaling function must be a scaling function, by Theorem 2.1, we get (vi). Combining (v) with (vi), we get (vii).

Conversely, suppose that G is a bounded closed set and (i)–(iii) and (vii) hold. Since (vii) implies (vi), by Theorem 2.1, we know that there exists a scaling function φ with $\operatorname{supp}\hat{\varphi} = G$. Again by (vii) and Lemma 3.2, φ is a W-type scaling function. □

3.2. Convex supports

Here we discuss the case that the supports of Fourier transforms of scaling functions are convex sets.

Theorem 3.3. *Let φ be a scaling function with* $\operatorname{supp}\hat{\varphi} = G$. *If G is a convex set, then*

(a) $0 \in G^o$ *(0 is an interior point of G),*
(b) *φ is a W-type scaling function.*

Proof. Since φ is a band-limited scaling function with $\operatorname{supp}\hat{\varphi} = G$, by Theorem 2.1, (i)–(iii) and (vi) hold.

By (ii), we can take a point ω_1 lying in the first quadrant: $\{\omega = (\omega_1, \ldots, \omega_d)\colon\ \omega_i > 0\ (i = 1, \ldots, d)\}$ of R^d such that $\omega_1 \in \bigcup_m 2^m G$, further there exists $m_1 \in Z$ such that $\omega_1 \in 2^{m_1}G$. Denote $b^{(1)} := 2^{-m_1}\omega_1$. So the point $b^{(1)} \in G$

and lies in the first quadrant of R^d. Similarly, there are the points $b^{(l)} \in G$ ($l = 1, \dots, 2^d$) lying in every quadrant of R^d, respectively. Let Q be the polyhedron with the 2^d vertexes $\{b^{(l)}\}_{l=1}^{2^d}$. Then $0 \in Q^o$. Since G is a convex set and $\{b^{(l)}\}_{l=1}^{2^d} \in G$, we see that $Q \subset G$. So $0 \in G^o$. We get (a).

By $|\partial G| = 0$ and Theorem 2.11(c), we have

$$\left(G^o \Big\backslash \frac{1}{2}G\right) \cap \left(\frac{1}{2}G^o + 2\pi\nu\right) = \emptyset, \quad \nu \in Z^d. \tag{3.5}$$

For any $\omega^* \in \frac{1}{2}G^o$ and any $\nu^* \in Z^d \setminus \{0\}$, take the straight line through ω^*

$$L\colon\ \omega = \omega(\lambda) = \omega^* + 2\pi\nu^*\lambda \quad (-\infty < \lambda < \infty). \tag{3.6}$$

By $\frac{1}{2}G \subset G$, we have $\omega^* \in G^o$. Again since G is a bounded convex set, we see that the straight line L intersects ∂G at two points $\omega_1 := \omega(\lambda_1)$ and $\omega_4 := \omega(\lambda_4)$ ($\lambda_1 < \lambda_4$) and the straight line L intersects $\partial(\frac{1}{2}G)$ at two points $\omega_2 := \omega(\lambda_2)$ and $\omega_3 := \omega(\lambda_3)$ ($\lambda_2 < \lambda_3$). Since G is convex and $0 \in G^o$, we have $\partial G \cap \partial(\frac{1}{2}G) = \emptyset$, $0 \in \frac{1}{2}G^o$. So $\lambda_1 < \lambda_2$ and $\lambda_3 < \lambda_4$. Again by $\omega^* = \omega(0)$, we see that

$$\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4. \tag{3.7}$$

By $A(\alpha, \beta)$ denote the open line segment connecting the points α and β except two end points. Since $A(\omega_3, \omega_4) \subset G^o \setminus \frac{1}{2}G$ and $A(\omega_2, \omega_3) \subset \frac{1}{2}G^o$, by (3.5), we conclude that

$$A(\omega_3, \omega_4) \cap \big(A(\omega_2, \omega_3) + 2\pi\nu^*\big) = \emptyset. \tag{3.8}$$

From this, we can prove that

$$G^o \cap \big(A(\omega_2, \omega_3) + 2\pi\nu^*\big) = \emptyset. \tag{3.9}$$

In fact, by (3.6) and $\omega_i = \omega(\lambda_i)$, we have $\omega_i + 2\pi\nu^* = \omega^* + 2\pi\nu^*(\lambda_i + 1) = \omega(\lambda_i + 1)$ ($i = 2, 3$). From this and

$$A(\omega_2, \omega_3) + 2\pi\nu^* = A(\omega_2 + 2\pi\nu^*, \omega_3 + 2\pi\nu^*),$$

it follows by (3.8) that

$$A\big(\omega(\lambda_3), \omega(\lambda_4)\big) \cap A\big(\omega(\lambda_2 + 1), \omega(\lambda_3 + 1)\big) = \emptyset.$$

Again by (3.7), we see that this is equivalent to

$$\lambda_2 + 1 \geqslant \lambda_4. \tag{3.10}$$

Now if $\zeta = \omega(\tilde{\lambda}) \in A(\omega_2, \omega_3) + 2\pi\nu^*$, then $\lambda_2 + 1 \leqslant \tilde{\lambda} \leqslant \lambda_3 + 1$. From this and (3.10), we have $\tilde{\lambda} \geqslant \lambda_4$. Since $\omega_4 = \omega(\lambda_4) \in \partial G \cap L$, we see that $\zeta \notin G^o$, i.e., (3.9) holds.

By (3.7), $\omega^* \in A(\omega_2, \omega_3)$. From this and (3.9), it follows that $\omega^* + 2\pi\nu^* \notin G^o$. Noticing that ω^* is any point in $\frac{1}{2}G^o$ and ν^* is any point in $Z^d \setminus \{0\}$, we have

$$G^o \cap \left(\frac{1}{2}G^o + 2\pi\nu\right) = \emptyset \quad (\nu \neq 0).$$

Again by $|\partial G| = 0$, we get (vii). Using Lemma 3.2, φ is a W-type scaling function. □

Let G be the support of the Fourier transform of a band-limited scaling function. By Theorem 2.11, we know that $0 \in G$. Especially, if G is convex, by Theorem 3.3, we know that $0 \in G^o$. It is natural to ask whether there exists a scaling function φ such that $0 \in \partial G$. The answer is positive.

Example 3.4. Let $G \subset R^2$ and

$$G = \big([-\pi, \pi]^2 \setminus E^0\big) \cup (E - 2\pi e_1),$$

where $E = [0, \pi]^2 \setminus D(\pi e_1, \pi)$ and $e_1 = (1, 0)$. Let $\hat{\varphi} = X_G$. Then $\operatorname{supp}\hat{\varphi} = G$ and it is easily check that (i)–(iii) and (iv) hold. By Madych's theorem, φ is an M-type scaling function for $L^2(R^2)$. Here $0 \in \partial G$.

If G is bounded convex and completely symmetric about the origin, we have

Theorem 3.5. *Let* φ *be a band-limited scaling function with* $\operatorname{supp}\hat{\varphi} = G$. *If* G *is convex and completely symmetric about the origin, then*

$$[-\pi, \pi]^d \subset G \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d. \tag{3.11}$$

Proof. Let $e_k = (\mu_1, \ldots, \mu_d)$, where $\mu_j = 0$ $(j \neq k)$ and $\mu_k = 1$. Take a straight line through the origin

$$L_k\colon \omega = \lambda e_k \quad (-\infty < \lambda < \infty),\ k = 1, \ldots, d. \tag{3.12}$$

By Theorem 3.3, $0 \in G^o$, so the line L_k intersects ∂G at two points ξ^k and η^k. Denote

$$\xi^k = x_k e_k, \qquad \eta^k = y_k e_k \quad (x_k, y_k \in R). \tag{3.13}$$

Without loss of generality, we assume $y_k > 0$. Since G is completely symmetric about the origin, we have $x_k = -y_k$, $\xi^k = -\eta^k$, and

$$-\eta^k, \eta^k \in L_k \cap \partial G, \qquad -\frac{1}{2}\eta^k, \frac{1}{2}\eta^k \in L_k \cap \partial\left(\frac{1}{2}G\right). \tag{3.14}$$

By Theorem 3.3(b), we know that φ is a W-type scaling function. Hence, by Theorem 3.1,

$$G \cap \left(\frac{1}{2}G + 2\pi\nu\right) \simeq \emptyset \quad (\nu \neq 0).$$

Similar to the argument of Theorem 2.11(c), we can conclude from $|\partial G| = 0$ that

$$G^o \cap \left(\frac{1}{2}G^o + 2\pi\nu\right) = \emptyset \quad (\nu \neq 0).$$

For each k, from this and (3.14), we have

$$A\left(-\eta^k, \eta^k\right) \cap \left(\frac{1}{2}A\left(-\eta^k, \eta^k\right) + 2\pi e_k\right) = \emptyset,$$

where $A(\alpha, \beta)$ is stated in Theorem 3.3. Again by (3.13), we get

$$A(-y_k e_k, y_k e_k) \cap A\left(\left(-\frac{1}{2}y_k + 2\pi\right)e_k,\ \left(\frac{1}{2}y_k + 2\pi\right)e_k\right) = \emptyset.$$

This implies $-\frac{1}{2}y_k + 2\pi \geqslant y_k$. So we have

$$y_k \leqslant \frac{4}{3}\pi. \tag{3.15}$$

For any $\zeta = \sum_{j=1}^{d} \zeta_j e_j \in G$, take $\tau = \zeta_k e_k - \sum_{j \neq k} \zeta_j e_j$. Since G is completely symmetric about the origin, $\tau \in G$. Again since G is convex, $\frac{1}{2}(\zeta + \tau) = \zeta_k e_k \in G$. By (3.12)–(3.14), we have

$$\zeta_k e_k \in L_k \cap G, \quad -y_k e_k, y_k e_k \in L_k \cap \partial G.$$

So $-y_k \leqslant \zeta_k \leqslant y_k$. From this and (3.15), we have

$$-\frac{4}{3}\pi \leqslant \zeta_k \leqslant \frac{4}{3}\pi.$$

Since k is arbitrary, we have $\zeta \in \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d$. Namely, $G \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d$.

Below we prove $[-\pi, \pi]^d \subset G$. Since G is convex, we know that if all vertexes of $[-\pi, \pi]^d$ lie in G, then $[-\pi, \pi]^d \subset G$. Hence it is enough to prove that all vertexes of $[-\pi, \pi]^d$ lie in G.

Suppose that there exists a vertex p of $[-\pi, \pi]^d$ such that $p \notin G$. Since G is completely symmetric about the origin, we know that all vertexes of $[-\pi, \pi]^d$ do not belong to G. It is clear that for any $\nu \in Z^d$, the point $p + 2\pi\nu$ either is still a vertex of $[-\pi, \pi]^d$ or lies outside $\left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d$. If the first case occurs, since all vertexes do not belong to G, we have $p + 2\pi\nu \notin G$. If the second case occurs, by $G \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^d$, we have also $p + 2\pi\nu \notin G$. Therefore, we always have $p+2\pi\nu\notin G$ ($\nu\in Z^d$). This is equivalent to $p\notin G+2\pi Z^d$. However, since φ is band-limited, by Theorem 2.11(b), we have $G+2\pi Z^d=R^d$. This is a contradiction. So all vertexes of $[-\pi,\pi]^d$ lie in G. $\square$

Since the support of the Fourier transform of Shannon's scaling function is $[-\pi,\pi]$ and the support of the Fourier transform of Meyer's scaling function is $\left[-\frac{4}{3}\pi,\frac{4}{3}\pi\right]$, (3.11) cannot be improved.

Conversely, if a closed set G satisfies (3.11), then $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]^d\subset\frac{1}{2}G\subset\left[-\frac{2}{3}\pi,\frac{2}{3}\pi\right]^d$, so (i) and (vii) hold. Again by $[-\pi,\pi]^d\subset G$, we have (ii) and (iii) hold. Applying Theorem 3.1, we have

Corollary 3.6. *If a closed set G satisfies* (3.11)*, then there is a W-type scaling function* φ *with* $\operatorname{supp}\hat{\varphi}=G$.

4. Can the support of $\hat{\varphi}$ be a closed ball?

We know that the support of Meyer's scaling function is a closed interval, which is a closed ball in the one-dimensional case, however, for $d>1$, we have

Theorem 4.1. *The support of the Fourier transform of any scaling function of* $L^2(R^d)$ *cannot be a closed ball in* R^d ($d>1$).

Proof. If it is not true, then there is a scaling function φ such that $\operatorname{supp}\hat{\varphi}=\bar{D}(a,r)$. By Theorem 2.11(b), we have

$$\bigcup_{\nu\in Z^d}\big(\bar{D}(a,r)+2\nu\pi\big)=R^d. \tag{4.1}$$

From this, we can conclude that

$$r\geqslant\pi\sqrt{d}. \tag{4.2}$$

If (4.2) is not true, then $2\pi>\frac{2r}{\sqrt{d}}$. Hence there exists $\epsilon>0$ such that

$$2\pi>\frac{2(r+\epsilon)}{\sqrt{d}}. \tag{4.3}$$

Let $\sigma=(1,\dots,1)$. Take the points

$$p_\nu=a-\pi(2\nu-\sigma)\quad\big(\nu\in Z^d\big). \tag{4.4}$$

Let $\nu=(\eta_1,\dots,\eta_d)\in Z^d$. Since η_i is an integer, $|2\eta_i-1|\geqslant 1$, further, we have

$$|2\nu-\sigma|=\left(\sum_{i=1}^{d}(2\eta_i-1)^2\right)^{\frac{1}{2}}\geqslant\sqrt{d}\quad\big(\nu\in Z^d\big).$$

Again by (4.3) and (4.4), we have $|p_\nu-a|>r+\epsilon$, and so $\bar{D}(p_\nu,\epsilon)\cap\bar{D}(a,r)=\emptyset$. Noticing that $p_\nu=p_0-2\pi\nu$, we have $\bar{D}(p_0,\epsilon)\cap(\bar{D}(a,r)+2\pi\nu)=\emptyset$ ($\nu\in Z^d$). Hence

$$\bar{D}(p_0,\epsilon)\cap\left(\bigcup_{\nu\in Z^d}\big(\bar{D}(a,r)+2\pi\nu\big)\right)=\emptyset,$$

this is contrary to (4.1). Hence (4.2) holds.

On the other hand, since $\operatorname{supp}\hat{\varphi}=\bar{D}(a,r)$ is convex, by Theorem 3.3(b), we know that φ is a W-type scaling function. Again by Theorem 3.1, (vii) holds. So

$$\bar{D}(a,r)\cap\bar{D}\left(\frac{a}{2}+2\pi\nu,\frac{r}{2}\right)\simeq\emptyset\quad(\nu\neq 0).$$

From this, we can see that

$$\left|\frac{a}{2}-2\pi\nu\right|\geqslant\frac{3}{2}r\quad(\nu\neq 0). \tag{4.5}$$

Let $a = (a_1, \dots, a_d)$ and $|a_k| = \max\{|a_1|, \dots, |a_d|\}$. Denote $\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_d^2}$. Then $\|a\|^2 \leqslant d|a_k|^2$. Set

$$\mu(a_k) = \begin{cases} 1, & a_k \geqslant 0, \\ -1, & a_k < 0, \end{cases}$$

we have

$$\left\| \frac{a}{2} - 2\pi\mu(a_k)e_k \right\|^2 = \sum_{i \neq k} \left| \frac{a_i}{2} \right|^2 + \left(\frac{a_k}{2} - 2\pi\mu(a_k) \right)^2 = \frac{1}{4}\|a\|^2 - 2\pi|a_k| + 4\pi^2, \tag{4.6}$$

where e_k is stated in Theorem 3.5. From this and $\|a\|^2 \leqslant d|a_k|^2$, we get

$$\left\| \frac{a}{2} - 2\pi\mu(a_k)e_k \right\|^2 \leqslant g\big(\|a\|\big), \tag{4.7}$$

where

$$g(x) = \frac{1}{4}x^2 - \frac{2\pi}{\sqrt{d}}x + 4\pi^2. \tag{4.8}$$

Since $\operatorname{supp}\hat{\varphi} = \bar{D}(a, r)$ is convex, by Theorem 3.3(a), the origin is an interior point of $\bar{D}(a, r)$, so $\|a\| < r$. From this and (4.8), we have $g(\|a\|) \leqslant \max\{g(0), g(r)\}$. Again by (4.7) and (4.8), we get

$$\left\| \frac{a}{2} - 2\pi\mu(a_k)e_k \right\|^2 \leqslant \max\left\{ 4\pi^2, \frac{1}{4}r^2 - \frac{2\pi}{\sqrt{d}}r + 4\pi^2 \right\}. \tag{4.9}$$

Noticing that $\mu(a_k)e_k \in Z^d \setminus \{0\}$, by (4.5) and (4.9), we conclude that either $r \leqslant \frac{4}{3}\pi$ or

$$r^2 + \frac{\pi}{\sqrt{d}}r \leqslant 2\pi^2. \tag{4.10}$$

Since $d > 1$, it is clear that $r \leqslant \frac{4}{3}\pi$ is contrary to (4.2). By (4.10) and $d > 1$, we have

$$r^2 + \frac{\pi}{\sqrt{d}}r < (\pi\sqrt{d})^2 + \frac{\pi}{\sqrt{d}} \cdot \pi\sqrt{d},$$

this is equivalent to $r < \pi\sqrt{d}$. This is also contrary to (4.2).

Therefore, for $d > 1$, there does not exist any scaling function such that the support of its Fourier transform is a closed ball in R^d. □

Acknowledgments

The author thanks the editor and the referees for valuable comments and suggestions.

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