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A Characterization of Generalized Frame MRAs Deriving Orthonormal Wavelets

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Abstract In 2000, Papadakis announced that any orthonormal wavelet must be derived by a generalized frame MRA (GFMRA). In this paper, we give a characterization of GFMRAs which can derive orthonormal wavelets, and show a general approach to the constructions of non-MRA wavelets. Finally we present two examples to illustrate the theory.

Keywords orthonormal wavelets, generalized frame MRA, non-MRA wavelets
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1 Introduction

In order to construct orthonormal wavelets, Mallat [1] established the theory of the classical multiresolution analysis (MRA). However, indeed there exist orthonormal wavelets which can not be derived by MRAs, one calls them non-MRA wavelets. The constructions of non-MRA wavelets are very difficult. Until now, except the Journé wavelet [2], there are only several examples of non-MRA wavelets constructed by Bownik, et al. [3].

In 2000, Papadakis [4] introduced the concept of a generalized frame MRA (GFMRA) and announced that any orthonormal wavelet with a single mother function is derived by a GFMRA [4]. This is a very important result in wavelet analysis.

Conversely, it is natural to ask whether every GFMRA can derive orthonormal wavelets. In Section 5, we present a counter-example (Example 2). So two problems are raised. The first problem is under which condition a GFMRA can derive an orthonormal wavelet. In Sections 3–4, we give a necessary and sufficient condition (Theorem 1), and we discuss it in the extensive case of many mother functions. In addition, Theorem 1 is also a generalization of the classical result in the theory of MRA. The second problem, which is related closely to the first problem, is how to construct non-MRA wavelets. In the proof of Theorem 1, based on GFMRAs, we construct orthonormal wavelets (of course, including non-MRA wavelets). Combining this with Papadakis' announcement, we have solved the problem of constructing all non-MRA wavelets. Finally, we present an example (Example 1) about constructing a non-MRA wavelet from a GFMRA with two scaling functions.

2 Preliminaries

We will list some notations, definitions, propositions and theorems.

All through this paper, $\bigoplus$ expresses the orthogonal sum and "span" expresses the linear combination. For convenience, we use the notations: $(\,a_k\,)_k = (\,a_k\,)_{k=-\infty}^{\infty}$ and $\sum_k = \sum_{k=-\infty}^{\infty}$. The notation "a.e." is often omitted.

If $f \in L^1(R)$, we define the Fourier transform $\widehat{f}(\omega) = \int_R f(t)e^{-it\omega}dt$, and as usual, extend it to an operator from $L^2(R)$ to $L^2(R)$.

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Denote the bracket product $[\widehat{f}\,,\,\widehat{g}](\omega)=\sum_k \widehat{f}(\omega+2k\pi)\overline{\widehat{g}}(\omega+2k\pi)$. It is well known that if $f,g\in L^2(R)$, then the series $\sum_k \widehat{f}(\omega+2k\pi)\overline{\widehat{g}}(\omega+2k\pi)$ is convergent absolutely for a. e. $\omega\in$ R.

Let $\{g_l\}_1^r\subset L^2(R)$. Denote the subspace of $L^2(R)$: $S=S(\{g_l\}_1^r)=\overline{\text{span}}\{\,g_l(t-n),\ n\in Z,\ l=1,\dots,r\}$ and the corresponding subspace of l^2 : $J_S(\omega)=\{\,(\ \widehat{f}(\omega+2k\pi)\ )_k,\ f\in S\,\}$ for a.e. $\omega\in$ R.

The sequence $\{f_n\}$ is a frame for a subspace U of $L^2(R)$ if there exist $A,B>0$ such that

$$A\parallel f\parallel^2\le\sum_n|(f,f_n)|^2\le B\parallel f\parallel^2,\quad \forall f\in U.$$

The following propositions are often used in this paper:

Proposition 1 [5, p. 200] *Let* $S=S(\,\{g_l\}_1^r\,)$. *Then*: $J_S(\omega)=\text{span}\{\,(\ \widehat{g}_l(\omega+2k\pi)\ )_k,\ l=1,\dots,r\}$.

Proposition 2 [5] *Let* $f,g\in L^2(R)$. *Then*:

(i) $S(\{f\})$ *and* $S(\{g\})$ *are orthogonal if and only if* $[\widehat{f},\widehat{g}](\omega)=0$ *a. e.* $\omega\in R$;

(ii) $\{f(t-n),\ n\in Z\}$ *is a frame for* $S(\{f\})$ *if and only if there exist* $A,B>0$ *such that*

$$A\le[\widehat{f},\widehat{f}](\omega)\le B\qquad a.e.\ on\ \{\omega:[\widehat{f},\widehat{f}](\omega)\ne0\};$$

(iii) $\{f(t-n),\ n\in Z\}$ *is an orthonormal system if and only if* $[\widehat{f},\widehat{f}](\omega)=1$ *a.e.* $\omega\in R$.

Proposition 3 [5, p. 200] *If* $\Phi=\{\varphi^{(j)}\}_1^r\subset L^2(R)$, *then there exists a family of r functions* $\Psi=\{\psi^{(j)}\}_1^r\subset L^2(R)$ *such that* $S(\,\Phi\,)=S(\Psi)$ *and for all* j,l,

$$[\widehat{\psi}^{(j)},\widehat{\psi}^{(l)}](\omega)=\begin{cases}1\ or\ 0, & j=l,\\ 0, & j\ne l,\end{cases}\qquad a.e.\ \omega\in R.$$

Hereafter "$\widehat{f}(\omega)=1$ or 0 a.e. $\omega\in R$" means that either $\widehat{f}(\omega)=1$ or $\widehat{f}(\omega)=0$ for a fixed a.e. $\omega\in R$.

Below we give a equivalent definition of GFMRA.

Definition 1 *Let* $\{V_m\}_{m\in Z}$ *be a sequence of the closed subspaces of* $L^2(R)$ *and satisfy the following*:

(a) $V_m\subset V_{m+1},\ m\in Z,\ \overline{\bigcup_m V_m}=L^2(R),\ \bigcap_m V_m=\{0\}$;

(b) $f\in V_m\leftrightarrow f(2\cdot)\in V_{m+1},\ m\in Z$;

(c) *There exists a family of functions* $\{\varphi^{(j)}\}_1^r\subset V_0$ *such that* $V_0=S(\,\{\varphi^{(j)}\}_1^r\,)$ *and for all* j,l,

$$[\widehat{\varphi}^{(j)},\widehat{\varphi}^{(l)}](\omega)=\begin{cases}1\ or\ 0, & j=l,\\ 0, & j\ne l\end{cases}\qquad a.e.\ \omega\in R. \tag{2.1}$$

Then $\{V_m\}_{m\in Z}$ *is called a generalized frame multiresolution analysis* (GFMRA), *and* $\{\varphi^{(j)}\}_1^r$ *are called scaling functions.*

Remark 1 In the condition (c) of Definition 1, if $r=1$ and $[\widehat{\varphi}\,,\,\widehat{\varphi}](\omega)=1$ a.e. $\omega\in R$, the GFMRA reduces to the classical MRA.

Remark 2 In the definition of GFMRA given by Papadakis, the condition (c) is replaced by the following condition:

(c*) *"There exists* $\{\varphi^{(j)}\}_1^r$ *in* V_0 *such that the set* $\{\varphi^{(j)}(t-n),\ j=1,\dots,r\ n\in Z\}$ *is a frame for* V_0*".*

It is not difficult to see that Definition 1 and Papadakis' definition [4] of GFMRA are equivalent. In fact, by Proposition 2(i), (ii), we see that the condition (c) implies the condition (c*). Conversely, by Proposition 3, we know that the condition (c*) implies the condition (c).

The sequence $\{\psi^{(j)}\}_1^p$ is an orthonormal wavelet with p mother functions if $\{\psi_{m,n}^{(j)}:=2^{\frac{m}{2}}\psi^{(j)}(2^mt-n),\ \ m,n\in Z,\ \ j=1,\dots,p\}$ is an orthonormal basis for $L^2(R)$.

Let $\{V_m\}$ be a GFMRA and $\{\psi^{(j)}\}_1^p$ an orthonormal wavelet. If $\{\psi^{(j)}(t-n),\ n\in Z,\ j=1,\dots,p\}$ is an orthonormal basis of W_0 (where $V_1=V_0\bigoplus W_0$), then we say that $\{\psi^{(j)}\}_1^p$ is an orthonormal wavelet associated with the GFMRA $\{V_m\}$, or we say that an orthonormal wavelet $\{\psi^{(j)}\}_1^p$ is derived by the GFMRA $\{V_m\}$.

The following results are important in wavelet analysis:

Mallat's Theorem [1] *Any MRA can derive an orthonormal wavelet with a single mother function.*

Papadakis' Theorem [4] *Any orthonormal wavelet with a single mother function is derived by a GFMRA.*

3 Main Theorem

In 1989, Mallat [1] showed that any MRA can derive an orthonormal wavelet. Later on, Journé [2] showed further that not all orthonormal wavelets can be derived by MRAs. However, in 2000, Papadakis [4] announced that all orthonormal wavelets can be derived by GFMRAs. In the present paper, we consider the inverse problem of Papadakis' theorem, that is, does a GFMRA certainly derive an orthonormal wavelet? The answer is negative. Example 2 in Section 5 is a counter-example. Moreover, we show in Theorem 1 that under which condition a GFMRA can derive an orthonormal wavelet. Meanwhile, in the proof of Theorem 1, we also present a general approach to the constructions of orthonormal wavelets.

Theorem 1 *Let $\{V_m\}$ be a GFMRA with r scaling functions $\{\varphi^{(j)}\}_1^r$. Then this GFMRA can derive an orthonormal wavelet with p mother functions if and only if*

$$\tau\left(\frac{\omega}{2}+\pi\right)+\tau\left(\frac{\omega}{2}\right)-\tau(\omega)=p \quad a.e.\ \omega\in R, \tag{3.1}$$

where

$$\tau(\omega)=\sum_{j=1}^{r}[\widehat{\varphi}^{(j)},\widehat{\varphi}^{(j)}](\omega). \tag{3.2}$$

Papadakis shows that any orthonormal wavelet with a single mother function is derived by a GFMRA. Conversely, by Theorem 1, we get the following:

Corollary 1 *A GFMRA can derive an orthonormal wavelet with a single mother function if and only if $\tau(\frac{\omega}{2}+\pi)+\tau(\frac{\omega}{2})-\tau(\omega)=1$ a.e. $\omega\in R$.*

In the proof of Theorem 1, we present a general approach to the constructions of the orthonormal wavelets from GFMRAs. Again by Papadakis' theorem, the problem of constructing all non-MRA wavelets has been solved.

For an MRA, the scaling function satisfies $[\widehat{\varphi},\widehat{\varphi}](\omega)=1$ a.e. $\omega\in R$, so

$$\tau\left(\frac{\omega}{2}+\pi\right)+\tau\left(\frac{\omega}{2}\right)-\tau(\omega)=1 \ \text{ a.e. } \omega\in R.$$

Again by Corollary 1, we obtain Mallat's classical result: "Any MRA can derive an orthonormal wavelet with a single mother function."

4 Proof of Theorem 1

To prove Theorem 1, we need the following lemma:

Lemma 1 *Let $\{V_m\}$ be a GFMRA with r scaling functions $\{\varphi^{(j)}\}_1^r$. Then Formula* (3.1) *is equivalent to* $\dim J_{V_1}(\omega)-\dim J_{V_0}(\omega)=p$ *a.e.* $\omega\in R$.

Proof Since $\{\varphi^{(j)}\}_1^r$ are scaling functions, $V_0=S(\{\varphi^{(j)}\}_1^r)$ and (2.1) holds. By $V_0=S(\{\varphi^{(j)}\}_1^r)$ and Proposition 1, it follows that

$$J_{V_0}(\omega)=\operatorname{span}\{(\widehat{\varphi}^{(j)}(\omega+2k\pi))_k,\ j=1,\ldots,r\}. \tag{4.1}$$

By (2.1) and the definition of the bracket product, we have

$$[\widehat{\varphi}^{(j)},\widehat{\varphi}^{(l)}](\omega)=\sum_k\widehat{\varphi}^{(j)}(\omega+2k\pi)\overline{\widehat{\varphi}^{(l)}(\omega+2k\pi)}=0 \quad \text{a.e. } \omega\in R\ (j\neq l), \tag{4.2}$$

and

$$[\widehat{\varphi}^{(j)}, \widehat{\varphi}^{(j)}](\omega) = \sum_k |\widehat{\varphi}^{(j)}(\omega + 2k\pi)|^2 = 1 \text{ or } 0 \quad \text{a.e. } \omega \in R.$$

From this and (4.2), we can see that the vectors $\{(\, \widehat{\varphi}^{(j)}(\omega + 2k\pi)\,)_k,\ j = 1, \dots, r\}$ in l^2 are orthogonal to each other, and the square of the norm of the vector $(\, \widehat{\varphi}^{(j)}(\omega + 2k\pi)\,)_k$ is $\left[\widehat{\varphi}^{(j)}, \widehat{\varphi}^{(j)}\right](\omega)$. Furthermore, by (4.1),

$$\dim J_{V_0}(\omega) = \dim(\text{span}\{(\widehat{\varphi}^{(j)}(\omega + 2k\pi))_k,\ j = 1, \dots, r\}) = \sum_{j=1}^{r} [\widehat{\varphi}^{(j)}, \widehat{\varphi}^{(j)}](\omega). \tag{4.3}$$

So, by (3.2), we have

$$\dim J_{V_0}(\omega) = \tau(\omega). \tag{4.4}$$

Let $\varphi_0^{(j)}(t) = \varphi^{(j)}(2t),\ \varphi_1^{(j)}(t) = \varphi^{(j)}(2t - 1)$. In view of $V_1 = \overline{\text{span}}\{\varphi^{(j)}(2t - k),\ k \in Z, j = 1, \dots, r\}$, we have

$$V_1 = S(\, \{\varphi_0^{(j)},\ \varphi_1^{(j)}\}_1^r\,). \tag{4.5}$$

Again letting $h_{j,k}(\omega) = \widehat{\varphi}^{(j)}(\frac{\omega}{2} + k\pi)$, by Proposition 1, we obtain that, for a fixed a.e. $\omega \in R$,

$$\begin{aligned} J_{V_1}(\omega) &= \text{span}\left\{ \left(\frac{1}{2} h_{j,k}(\omega) \right)_k,\ \left(\frac{1}{2} e^{-i\frac{\omega}{2}} (-1)^k h_{j,k}(\omega) \right)_k,\ j = 1, \dots, r \right\} \\ &= \text{span}\{\ (\, h_{j,k}(\omega)\,)_k\,,\ (\,(-1)^k h_{j,k}(\omega)\,)_k\,,\ j = 1, \dots, r\} \\ &= \text{span}\{A_j(\omega)\,,\ B_j(\omega),\ j = 1, \dots, r\}, \end{aligned}$$

where the vectors

$$A_j(\omega) = \left(\frac{1 + (-1)^k}{2} h_{j,k}(\omega) \right)_k, \quad B_j(\omega) = \left(\frac{1 - (-1)^k}{2} h_{j,k}(\omega) \right)_k$$

and the k-th components of the vectors $A_j(\omega)$ and $B_j(\omega)$ are as follows:

$$\begin{aligned} \frac{1 + (-1)^k}{2} h_{j,k}(\omega) &= \begin{cases} \widehat{\varphi}^{(j)}\left(\dfrac{\omega}{2} + k\pi \right), & k\text{-even}, \\ 0, & k\text{-odd}; \end{cases} \\ \frac{1 - (-1)^k}{2} h_{j,k}(\omega) &= \begin{cases} \widehat{\varphi}^{(j)}\left(\dfrac{\omega}{2} + k\pi \right), & k\text{-odd}, \\ 0, & k\text{-even}. \end{cases} \end{aligned} \tag{4.6}$$

From this, it follows that $A_j(\omega)$ and $B_l(\omega)$ are orthogonal in l^2 for $j, l = l, \dots, r$. Furthermore,

$$\begin{aligned} J_{V_1}(\omega) &= \text{span}\{A_j(\omega),\ j = 1, \dots, r\} \bigoplus \text{span}\{B_j(\omega),\ j = 1, \dots, r\} \\ &=: J_{V_1}^{(0)}(\omega) \bigoplus J_{V_1}^{(1)}(\omega) \quad \text{in } l^2. \end{aligned} \tag{4.7}$$

By (4.6), we see that the $2k$-th component of the vector $A_j(\omega)$ is just the k-th component of the vector $(\, \widehat{\varphi}^{(j)}(\frac{\omega}{2} + 2k\pi)\,)_k$, and the $(2k + 1)$-th component of the vector $A_j(\omega)$ is zero, so, by (4.7), we have

$$\dim J_{V_1}^{(0)}(\omega) = \dim\{\text{span}\{A_j(\omega),\ j = 1, \dots, r\}\} = \dim\left\{ \text{span}\left(\widehat{\varphi}^{(j)}\left(\frac{\omega}{2} + 2k\pi \right)_k \right),\ j = 1, \dots, r \right\}.$$

Comparing the above formula with (4.3), it follows from (4.4) that $\dim J_{V_1}^{(0)}(\omega) = \dim J_{V_0}(\frac{\omega}{2}) = \tau(\frac{\omega}{2})$. Similarly, we get $\dim J_{V_1}^{(1)}(\omega) = \tau(\frac{\omega}{2} + \pi)$. Combining this with (4.7),

$$\dim J_{V_1}(\omega) = \dim J_{V_1}^{(0)}(\omega) + \dim J_{V_1}^{(1)}(\omega) = \tau\left(\frac{\omega}{2} \right) + \tau\left(\frac{\omega}{2} + \pi \right).$$

From this and (4.4), it follows that (3.1) is equivalent to $\dim J_{V_1}(\omega) - \dim J_{V_0}(\omega) = p$ a.e. $\omega \in R$.

Proof of Theorem 1 (i) We prove the "only if " part.

Suppose that $\Psi = \{\psi^{(j)}\}_1^p$ is an orthonormal wavelet derived by the GFMRA $\{V_m\}$. Then the family $\{\psi^{(j)}(t-n),\ n \in Z,\ j = 1, \ldots, p\}$ is an orthonormal basis for W_0 $(V_1 = V_0 \bigoplus W_0)$. By a well-known result [2],

$$\sum_k \widehat{\psi}^{(j)}(\omega + 2k\pi)\overline{\widehat{\psi}^{(l)}(\omega + 2k\pi)} = \begin{cases} 0, & j \neq l, \\ 1, & j = l, \end{cases} \quad \text{a.e. } \omega \in R.$$

From this and Proposition 1, it follows that $J_{W_0}(\omega) = \text{span}\{(\widehat{\psi}^{(j)}(\omega + 2k\pi))_k, j = 1, \ldots, p\}$ and $\dim J_{W_0}(\omega) = p$ a.e. $\omega \in R$. Since $V_1 = V_0 \bigoplus W_0$, we have $J_{V_1}(\omega) = J_{V_0}(\omega) \bigoplus J_{W_0}(\omega)$ in l^2 [7]. Furthermore, $\dim J_{V_1}(\omega) - \dim J_{V_0}(\omega) = \dim J_{W_0}(\omega) = p$. So, by Lemma 1, (3.1) follows.

(ii) We prove the "if " part.

Suppose that (3.1) holds. By (4.5) and $V_0 = S(\{\varphi^{(j)}\}_1^r) \subset V_1$, we have $V_1 = S(\{\varphi^{(j)}\}_1^r, \{\varphi_0^{(j)}, \varphi_1^{(j)}\}_1^r)$. Since the family of scaling functions $\{\varphi^{(j)}\}_1^r$ satisfies (2.1), using the method of argument of Proposition 3, we see that there exist $2r$ functions $\{\eta^{(j)}\}_1^{2r}$ such that $V_1 = S(\ \{\varphi^{(j)}\}_1^r, \{\eta^{(j)}\}_1^{2r}\ )$ and for $k, l = 1, \ldots, 2r,\ j = 1, \ldots, r$,

$$[\widehat{\eta}^{(k)}, \widehat{\varphi}^{(j)}](\omega) = 0, \qquad [\widehat{\eta}^{(k)}, \widehat{\eta}^{(l)}](\omega) = \begin{cases} 1 \text{ or } 0, & k = l, \\ 0, & k \neq l, \end{cases} \quad \text{a.e. } \omega \in R. \tag{4.8}$$

So, by Proposition 2(i) and $V_0 = S(\{\varphi^{(j)}\})_1^r$, it follows that

$$V_1 = S(\ \{\varphi^{(j)}\}_1^r\ ) \bigoplus S(\ \{\eta^{(j)}\}_1^{2r}\ ) = V_0 \bigoplus S(\{\eta^{(j)}\}_1^{2r}).$$

Again by $V_1 = V_0 \bigoplus W_0$, it is clear that

$$W_0 = S(\{\eta^{(j)}\}_1^{2r}). \tag{4.9}$$

So, by Proposition 1 and (4.8), using an argument similar to (4.3), we have

$$\dim J_{W_0}(\omega) = \dim(\text{span}\{(\widehat{\eta}^{(j)}(\omega + 2k\pi))_k,\ j = 1, \ldots, 2r\}) = \sum_{j=1}^{2r} [\widehat{\eta}^{(j)}, \widehat{\eta}^{(j)}](\omega).$$

But by (3.1) and Lemma 1, we see that

$$\dim J_{W_0}(\omega) = \dim J_{V_1}(\omega) - \dim J_{V_0}(\omega) = p, \quad \text{a.e. } \omega \in R. \tag{4.10}$$

So $p \geq 1$ and

$$\sum_{j=1}^{2r} [\widehat{\eta}^{(j)}, \widehat{\eta}^{(j)}](\omega) = p, \text{ a.e. } \omega \in R. \tag{4.11}$$

Starting from the $2r$ functions $\{\eta^{(j)}\}_1^{2r}$, we will construct directly an orthonormal wavelet with p mother functions, namely, we will construct p functions whose integer translates generate an orthonormal basis for W_0.

Define the point sets as follows:

$$F_0 := R, \quad F_j := \{\omega \in R \,|\, [\widehat{\eta}^{(j)}, \widehat{\eta}^{(j)}](\omega) = 0\}\ (j = 1, \ldots, 2r), \quad G_l := \bigcap_{j=0}^{l} F_j\ (l = 0, \ldots, 2r).$$

If $\omega \in G_{2r}$, then $\omega \in F_j$ $(j = 1, \ldots, 2r)$. So from the definition of F_j, we have

$$[\widehat{\eta}^{(j)}\ ,\ \widehat{\eta}^{(j)}](\omega) = 0\ (\omega \in F_j),\ j = 1, \ldots, 2r. \tag{4.12}$$

Furthermore, $\sum_{j=1}^{2r} [\widehat{\eta}^{(j)}\ ,\ \widehat{\eta}^{(j)}](\omega) = 0$ $(\omega \in G_{2r})$. So, by (4.11) and $p \geq 1$, we obtain that the measure of G_{2r} is zero.

Again define $2r$ functions $\{g_1^{(j)}\}_1^{2r}$ as follows.

For $j = 1$, let

$$\widehat{g}_1^{(1)}(\omega) = \widehat{\eta}^{(l)}(\omega),\ \omega \in G_{l-1} \setminus F_l \ \ (1 \leq l \leq 2r). \tag{4.13}$$

Noticing that $\bigcup_{l=1}^{2r}(G_{l-1} \setminus F_l) = \bigcup_{l=1}^{2r}(G_{l-1} \setminus G_l) = R \setminus G_{2r}$ (disjoint unions) and the measure of G_{2r} is zero, we know that $\widehat{g}_1^{(1)}(\omega)$ is defined on R almost everywhere. We also see that (4.13) is equivalent to

$$\widehat{g}_1^{(1)}(\omega)=\sum_{l=1}^{2r} I_{G_{l-1}\backslash F_l}(\omega)\widehat{\eta}^{(l)}(\omega),\ \omega\in R \tag{4.14}$$

(where $I_E(\omega)$ is the characteristic function of E).

For $2\le j\le 2r$, let

$$\widehat{g}_1^{(j)}(\omega)=\begin{cases}\widehat{\eta}^{(j)}(\omega), & \omega\in R\setminus G_{j-1},\\ 0, & \omega\in G_{j-1}.\end{cases} \tag{4.15}$$

Now we prove that

$$[\widehat{g}_1^{(1)},\widehat{g}_1^{(1)}](\omega)=1;\ [\widehat{g}_1^{(j)},\widehat{g}_1^{(l)}](\omega)=\begin{cases}1 \text{ or } 0, & j=l,\\ 0, & j\neq l,\end{cases}\quad (j,\,l=1,\dots,2r),\ \text{a.e. } \omega\in R. \tag{4.16}$$

From $F_l+2k\pi=F_l,\ G_l+2k\pi=G_l$ and (4.13), (4.15), it follows that, for $k\in Z$,

$$\widehat{g}_1^{(1)}(\omega+2k\pi)=\widehat{\eta}^{(l)}(\omega+2k\pi),\ \omega\in G_{l-1}\setminus F_l\quad (l=1,\dots,2r), \tag{4.17}$$

and

$$\widehat{g}_1^{(j)}(\omega+2k\pi)=\begin{cases}\widehat{\eta}^{(j)}(\omega+2k\pi), & \omega\in R\setminus G_{j-1},\\ 0, & \omega\in G_{j-1},\end{cases}\quad (j=2,\dots,2r). \tag{4.18}$$

So, by (4.17) and the definition of the bracket product, we have

$$[\widehat{g}_1^{(1)},\widehat{g}_1^{(1)}](\omega)=\sum_k|\widehat{g}_1^{(1)}(\omega+2k\pi)|^2=[\widehat{\eta}^{(l)},\widehat{\eta}^{(l)}](\omega),\ \omega\in G_{l-1}\setminus F_l\ \ (l=1,\dots 2r).$$

However, the definition of the point set F_l implies $\left[\widehat{\eta}^{(l)}\,,\,\widehat{\eta}^{(l)}\right](\omega)\neq 0,\ \omega\notin F_l$. Hence, by the second formula in (4.8), we have $\left[\widehat{\eta}^{(l)}\,,\,\widehat{\eta}^{(l)}\right](\omega)=1,\ \omega\in G_{l-1}\setminus F_l$. Again noticing that $\bigcup_{l=1}^{2r}(G_{l-1}\setminus F_l)=R\backslash G_{2r}$ and the measure of G_{2r} is zero, we get $[\widehat{g}_1^{(1)},\widehat{g}_1^{(1)}](\omega)=1$, a.e. $\omega\in R$, namely, the first formula in (4.16) holds.

Using the formula

$$\bigcup_{l=1}^{j-1}(G_{l-1}\setminus F_l)=R\setminus G_{j-1}, \tag{4.19}$$

we have $G_{l-1}\setminus F_l\subset R\setminus G_{j-1}(1\le l\le j-1)$. So, by (4.17), (4.18) and (4.8), it follows that

$$\begin{aligned}[\widehat{g}_1^{(1)},\widehat{g}_1^{(j)}](\omega)&=\sum_k\widehat{g}_1^{(1)}(\omega+2k\pi)\overline{\widehat{g}_1^{(j)}(\omega+2k\pi)}=\sum_k\widehat{\eta}^{(l)}(\omega+2k\pi)\overline{\widehat{\eta}^{(j)}(\omega+2k\pi)}\\ &=[\widehat{\eta}^{(l)},\widehat{\eta}^{(j)}](\omega)=0\qquad\text{for}\quad \omega\in G_{l-1}\backslash F_l\ \ (1\le l\le j-1).\end{aligned}$$

Again, by (4.19), we have $[\widehat{g}_1^{(1)},\widehat{g}_1^{(j)}](\omega)=0,\ \omega\in R\setminus G_{j-1}$. But, by (4.18), we know that $\widehat{g}_1^{(j)}(\omega+2k\pi)=0,\ \omega\in G_{j-1}, j=2,\dots,2r$. Hence

$$[\widehat{g}_1^{(1)},\widehat{g}_1^{(j)}](\omega)=0\quad\text{a.e. }\omega\in R\quad\text{for } j=2,\dots,2r.$$

For $j,\,l=2,\dots,2r$, by (4.8) and (4.18), it is clear that

$$[\widehat{g}_1^{(j)},\widehat{g}_1^{(l)}](\omega)=\begin{cases}1\text{ or }0, & j=l,\\ 0, & j\neq l,\end{cases}\qquad\text{a.e. }\omega\in R.$$

Combining the above results with the first formula in (4.16), we get the second formula in (4.16).

Below we prove

$$W_0=S(\{g_1^{(j)}\}_1^{2r}). \tag{4.20}$$

By (4.9) and the second formula in (4.8), using Proposition 2(i), (ii), we conclude that $\{\eta^{(l)}(t-n),\ n\in Z,\ l=1,\dots,2r\}$ is a frame for W_0. So, by a known result, we conclude that, for any $f \in W_0$, there exist the sequences $\{c_n^{(l)}\}_n \in l^2$ such that

$$f(t) = \sum_{l=1}^{2r} \sum_n c_n^{(l)} \eta^{(l)}(t-n) \quad (L^2).$$

Taking the Fourier transform, we get

$$\widehat{f}(\omega) = \sum_{l=1}^{2r} \sum_n c_n^{(l)} (e^{-in\omega} \widehat{\eta}^{(l)}(\omega)) = \sum_{l=1}^{2r} \Big(\sum_n c_n^{(l)} e^{-in\omega} \Big) \widehat{\eta}^{(l)}(\omega) = \sum_{l=1}^{2r} H_l(\omega) \widehat{\eta}^{(l)}(\omega), \tag{4.21}$$

where $H_l(\omega) = \sum_n C_n^{(l)} e^{-in\omega}$ $(L_{2\pi}^2)$. In the argument of (4.21), the Riesz theorem of the convergent subsequence is used.

Noticing that $R = (R \setminus G_{l-1}) \bigcup (G_{l-1} \setminus F_l) \bigcup G_l$ (disjoint union) for $l = 1, \dots, 2r$, we have

$$1 = I_{R \setminus G_{l-1}}(\omega) + I_{G_{l-1} \setminus F_l}(\omega) + I_{G_l}(\omega), \quad \omega \in R,$$

here $I_E(\omega)$ is the characteristic function of E. Furthermore, by (4.12) and $G_l \subset F_l$, we have $\widehat{\eta}^{(l)}(\omega) = 0$, $\omega \in G_l$. So, by (4.21), we get

$$\widehat{f}(\omega) = \sum_{l=1}^{2r} H_l(\omega) I_{G_{l-1} \setminus F_l}(\omega) \widehat{\eta}^{(l)}(\omega) + \sum_{l=2}^{2r} H_l(\omega) I_{R \setminus G_{l-1}}(\omega) \widehat{\eta}^{(l)}(\omega).$$

Again by (4.13) and (4.15), it follows that

$$\widehat{f}(\omega) = \sum_{l=1}^{2r} H_l^*(\omega) \widehat{g}_1^{(l)}(\omega), \tag{4.22}$$

where

$$H_1^*(\omega) = \sum_{l=1}^{2r} H_l(\omega) I_{G_{l-1} \setminus F_l}(\omega), \ \ H_l^*(\omega) = H_l(\omega) I_{R \setminus G_{l-1}}(\omega) \ \ (2 \le l \le 2r).$$

From $F_l + 2k\pi = F_l$, $G_{l-1} + 2k\pi = G_{l-1}$ $(k \in Z,\ 1 \le l \le 2r)$, we can obtain that $H_l^*(\omega) \in L_{2\pi}^2$ $(1 \le l \le 2r)$.

Let $H_l^*(\omega) = \sum_n \beta_n^{(l)} e^{-in\omega}$, $\{\beta_n^{(l)}\}_n \in l^2$. By (4.16) and Proposition 2(ii), we can conclude that, for every l, $\{g_1^{(l)}(t-n),\ n \in Z\}$ is a frame for their closed span. So, by $\{\beta_n^{(l)}\}_n \in l^2$, using a known result (Theorem 2.5 in [6]), we see that $\sum_n \beta_n^{(l)} g_1^{(l)}(t-n)$ is L^2-convergent. Furthermore, $\sum_n \beta_n^{(l)} (e^{-in\omega} \widehat{g}_1^{(l)}(\omega))$ is L^2-convergent. So, by the Riesz theorem of the convergent subsequence, we can obtain that

$$\sum_n \beta_n^{(l)} (e^{-in\omega} \widehat{g}_1^{(l)}(\omega)) = \Big(\sum_n \beta_n^{(l)} e^{-in\omega} \Big) \widehat{g}_1^{(l)}(\omega) = H_l^*(\omega) \widehat{g}_1^{(l)}(\omega). \tag{4.23}$$

But, by (4.22) and (4.23), we get $\widehat{f}(\omega) = \sum_{l=1}^{2r} \sum_n \beta_n^{(l)} \left(e^{-in\omega} \widehat{g}_1^{(l)}(\omega) \right)$ (L^2). So

$$f(t) = \sum_{l=1}^{2r} \sum_n \beta_n^{(l)} \hat{g}_1^{(l)}(t-n) \ \ (L^2). \tag{4.24}$$

Furthermore, for any $f \in W_0$, we have $f \in S(\{g_1^{(j)}\}_1^{2r})$. Therefore,

$$W_0 \subset S(\{g_1^{(j)}\}_1^{2r}). \tag{4.25}$$

On the other hand, since $(G_{l-1} \setminus F_l) + 2n\pi = G_{l-1} \setminus F_l$, we have $I_{G_{l-1} \setminus F_l}(\omega) \in L_{2\pi}^2$ and $I_{G_{l-1} \setminus F_l}(\omega) = \sum_n \gamma_n^{(l)} e^{-in\omega}$ $(\{\gamma_n^{(l)}\}_n \in l^2)$. Starting from (4.14), imitating the process from (4.22) to (4.24), we get

$$g_1^{(1)}(t) = \sum_{l=1}^{2r} \sum_n \gamma_n^{(l)} \eta^{(l)}(t-n) \ \ (L^2).$$

So $g_1^{(1)} \in S(\{\eta^{(l)}\}_1^{2r})$. Furthermore, by (4.9), $g_1^{(1)} \in W_0$.

Similarly, $g_1^{(j)} \in W_0$ $(j = 2, \dots, 2r)$, and so $S(\{g_1^{(j)}\}_1^{2r}) \subset W_0$. Combining this with (4.25), we conclude that $W_0 = S(\{g_1^{(j)}\}_1^{2r})$, i.e. (4.20) holds.

Denote $W_0^{(1)} = S(g_1^{(1)})$ and $U_0^{(1)} = S(\{g_1^{(j)}\}_2^{2r})$. Then, by (4.20) and (4.16), we have $W_0 = W_0^{(1)} \bigoplus U_0^{(1)}$; furthermore, by a known result [7],

$$J_{W_0}(\omega) = J_{W_0^{(1)}}(\omega) \bigoplus J_{U_0^{(1)}}(\omega) \quad \text{and} \quad \dim J_{W_0}(\omega) = \dim J_{W_0^{(1)}}(\omega) + \dim J_{U_0^{(1)}}(\omega). \tag{4.26}$$

By $W_0^{(1)} = S(g_1^{(1)})$ and Proposition 1, we have $J_{W_0^{(1)}}(\omega) = \mathrm{span}\{(\widehat{g}_1^{(1)}(\omega+2k\pi))_k\}$. But the first formula in (4.16) implies $\sum_k |\widehat{g}_1^{(1)}(\omega + 2k\pi)|^2 = 1$ a.e. $\omega \in R$. So $\dim J_{W_0^{(1)}}(\omega) = 1$ a.e. $\omega \in R$. Again, by (4.10) and (4.26), we have $\dim J_{U_0^{(1)}}(\omega) = p-1$ a.e. $\omega \in R$. Finally, by $W_0^{(1)} = S(g_1^{(1)})$ and the first formula in (4.16), using Proposition 2(iii), we obtain that $\{g_1^{(1)}(t-n),\ n \in Z\}$ is an orthonormal basis for $W_0^{(1)}$. Set $\psi^{(1)} = g_1^{(1)}$.

Now we start from $U_0^{(1)}$ and the family of functions $\{g_1^{(j)}\}_2^{2r}$. Once again, repeating the above process, we obtain two subspaces $W_0^{(2)}$, $U_0^{(2)}$ and a new family of functions $\{g_2^{(j)}\}_1^{2r-1}$ such that $U_0^{(1)} = W_0^{(2)} \bigoplus U_0^{(2)}$, where $W_0^{(2)} = S(g_2^{(1)})$, $U_0^{(2)} = S(\{g_2^{(j)}\}_2^{2r-1})$, and $\dim J_{U_0^{(2)}}(\omega) = p-2$ a.e. $\omega \in R$, and $\{g_2^{(1)}(t-n),\ n \in Z\}$ is an orthonormal basis for $W_0^{(2)}$. Set $\psi^{(2)} = g_2^{(1)}$.

Again, repeating the above process $p-2$ times, we can finally obtain p functions $\{\psi^{(j)}\}_1^p$ and $p+1$ subspaces $\{W_0^{(j)}\}_1^p$ and $U_0^{(p+1)}$ such that $\dim J_{U_0^{(p+1)}}(\omega) = 0$ (i.e. $U_0^{(p+1)} = \{0\}$), and

$$W_0 = W_0^{(1)} \bigoplus W_0^{(2)} \cdots \bigoplus W_0^{(p)} \quad (W_0^{(j)} = S(\psi^{(j)})),$$

and for every j, $\{\psi^{(j)}(t-n),\ n \in Z\}$ is an orthonormal basis for $W_0^{(j)}$. Furthermore the integer translates of p functions $\{\psi^{(j)}\}_1^p$ generate an orthonormal basis for W_0, so $\Psi = \{\psi^{(j)}\}_1^p$ is an orthonormal wavelet derived by the GFMRA $\{V_m\}$. Theorem 1 is proved.

5 Two Examples

Below we will construct a GFMRA with two scaling functions and give a non-MRA wavelet derived by it.

Example 1 Define two functions $\varphi_1(t)$ and $\varphi_2(t)$ satisfying $\widehat{\varphi}_1(\omega) = I_{E_1}(\omega)$, and $\widehat{\varphi}_2(\omega) = I_{E_2}(\omega)$, where $I_\Omega(\omega)$ is a characteristic function of Ω and

$$E_1 = \Big[-\frac{16\pi}{7}, -2\pi\Big] \bigcup \Big[-\frac{8\pi}{7}, -\pi\Big] \bigcup \Big[-\frac{4\pi}{7}, -\frac{2\pi}{7}\Big] \bigcup \Big[0, \frac{2\pi}{7}\Big], \quad E_2 = -E_1.$$

Clearly, the two functions $\{\varphi_1, \varphi_2\}$ satisfy the condition (2.1).

Define $V_0 = S(\{\varphi_1, \varphi_2\})$ and $V_m = \{f(t) \in L^2(R) \mid f(2^{-m}\cdot) \in V_0\}$ $(m \in Z)$. We will prove that $\{V_m\}$ is a GFMRA with two scaling functions $\{\varphi_1, \varphi_2\}$.

First we prove that

$$V_0 = \{f \in L^2(R) \mid \mathrm{supp}\widehat{f} \subset E\}, \tag{5.1}$$

where $E = E_1 \bigcup E_2$ and $\mathrm{supp}\widehat{f} = \mathrm{clos}\{\omega \in R, \widehat{f}(\omega) \neq 0\}$.

Suppose that $f \in L^2(R)$ and $\mathrm{supp}\widehat{f} \subset E$. Set

$$H_1(\omega) = \begin{cases} \widehat{f}(\omega), & \omega \in \Big(-\frac{16\pi}{7}, -\frac{2\pi}{7}\Big) \Big\backslash \Big(-2\pi, -\frac{12\pi}{7}\Big), \\ \widehat{f}(\omega + 2\pi), & \omega \in \Big(-2\pi, -\frac{12\pi}{7}\Big), \end{cases} \quad H_1(\omega+2n\pi) = H_1(\omega),\ n \in Z,$$

and

$$H_2(\omega) = \begin{cases} \widehat{f}(\omega), & \omega \in \Big(\frac{2\pi}{7}, \frac{16\pi}{7}\Big) \Big\backslash \Big(\frac{12\pi}{7}, 2\pi\Big), \\ \widehat{f}(\omega - 2\pi), & \omega \in \Big(\frac{12\pi}{7}, 2\pi\Big), \end{cases} \quad H_2(\omega + 2n\pi) = H_2(\omega),\ n \in Z.$$

Clearly, $H_1(\omega)$, $H_2(\omega) \in L^2_{2\pi}$. Furthermore, we can obtain that

$$\widehat{f}(\omega) = H_1(\omega)I_{E_1}(\omega) + H_2(\omega)I_{E_2}(\omega) = H_1(\omega)\widehat{\varphi}_1(\omega) + H_2(\omega)\widehat{\varphi}_2(\omega), \text{ a.e. } \omega \in R. \tag{5.2}$$

In fact, since $\mathrm{supp}\widehat{f} \subset E$ and $E = E_1 \bigcup E_2$, we need to prove only that the above formula holds on E. For $\omega \in (0, \frac{2\pi}{7})$, we have $\omega - 2\pi \in (-2\pi, -\frac{12\pi}{7})$. So, by the definition of $H_1(\omega)$, we see that $H_1(\omega) = H_1(\omega - 2\pi) = \widehat{f}(\omega)$, $\omega \in (0, \frac{2\pi}{7})$ and $I_{E_1}(\omega) = 1,\ I_{E_2}(\omega) = 0,\ \omega \in (0, \frac{2\pi}{7})$. So (5.2) holds on $(0, \frac{2\pi}{7})$. Similarly, we can prove that (5.2) holds on the rest of E. By (5.2), we obtain easily that $f \in V_0$. Conversely, if $f \in V_0$, then it is clear that $f \in L^2(R)$ and $\mathrm{supp}\widehat{f}(\omega) \subset E$. So (5.1) holds.

By (5.1), we get

$$V_1 = \{f \in L^2(R) \mid \mathrm{supp}\widehat{f} \subset 2E\}. \tag{5.3}$$

In view of $E \subset 2E$, we have $V_0 \subset V_1$, furthermore $V_m \subset V_{m+1},\ m \in Z$. From this and a known result (Theorem 2.3.8 in [5]), it follows that $\bigcap_m V_m = \{0\}$. Again, by the definition of E,

$$\bigcup_m 2^m E \supset \bigcup_m \Big[-\frac{2\pi}{7}2^m, \frac{2\pi}{7}2^m\Big].$$

So $\bigcup_m 2^m E = R$. Furthermore, by a known result (Proposition 2.3.4 in [5]), we obtain that $\overline{\bigcup_m V_m} = L^2(R)$.

To summarize all these results, by Definition 1, $\{V_m\}$ is a GFMRA with two scaling functions $\{\varphi_1, \varphi_2\}$.

Now we discuss whether this GFMRA can derive an orthonormal wavelet.

A direct calculation shows that

$$\tau(\omega) = [\widehat{\varphi}_1, \widehat{\varphi}_1](\omega) + [\widehat{\varphi}_2, \widehat{\varphi}_2](\omega) = \begin{cases} 1, & \omega \in \Big(-\pi, -\dfrac{6\pi}{7}\Big), \\ 0, & \omega \in \Big(-\dfrac{6\pi}{7}, -\dfrac{4\pi}{7}\Big), \\ 1, & \omega \in \Big(-\dfrac{4\pi}{7}, -\dfrac{2\pi}{7}\Big), \\ 2, & \omega \in \Big(-\dfrac{2\pi}{7}, 0\Big), \end{cases}$$

and $\tau(\omega) = \tau(-\omega),\ \tau(\omega + 2n\pi) = \tau(\omega),\ n \in Z$. From this we conclude easily that

$$\tau\Big(\frac{\omega}{2} + \pi\Big) + \tau\Big(\frac{\omega}{2}\Big) - \tau(\omega) = 1 \text{ a.e. } \omega \in R.$$

So, by Theorem 1, we know that this GFMRA can derive an orthonormal wavelet which has a single mother function.

Let W_0 be the orthogonal complement of V_0 in V_1. From (5.1) and (5.3), it is not difficult to see that $W_0 = \{f \in L^2(R) \mid \mathrm{supp}\widehat{f} \subset F\}$, where $F = [-\frac{32\pi}{7}, -4\pi] \bigcup [-\pi, -\frac{4\pi}{7}] \bigcup [\frac{4\pi}{7}, \pi] \bigcup [4\pi, \frac{32\pi}{7}]$ and the measure of F is 2π.

However, we know that the Journé wavelet $\psi(t)$ satisfies $\widehat{\psi}(\omega) = I_F(\omega)$, it is an orthonormal wavelet, and it is a non-MRA wavelet, see [2]. Now we will prove that the Journe wavelet $\psi(t)$ is derived by this GFMRA.

Let $f \in W_0$. Then $\widehat{f}(\omega) = 0,\ \omega \notin F$. Denote

$$H(\omega) = \begin{cases} \widehat{f}(\omega), & \omega \in \Big(-\pi, -\dfrac{4\pi}{7}\Big) \bigcup \Big(\dfrac{4\pi}{7}, \pi\Big), \\ \widehat{f}(\omega + 4\pi), & \omega \in \Big(0, \dfrac{4\pi}{7}\Big), \\ \widehat{f}(\omega - 4\pi), & \omega \in \Big(-\dfrac{4\pi}{7}, 0\Big), \end{cases} \quad \text{and } H(\omega + 2n\pi) = H(\omega),\ n \in Z.$$

It is easy to see that $\widehat{f}(\omega) = H(\omega)I_F(\omega) = H(\omega)\widehat{\psi}(\omega)$ a.e. and $H(\omega) \in L^2_{2\pi}$. Hence $W_0 = S(\psi)$. Furthermore, $\{\psi(t-n),\ n \in Z\}$ is an orthonormal basis in W_0. So the Journé wavelet ψ is an orthonormal wavelet derived by this GFMRA with two scaling functions $\{\varphi_1, \varphi_2\}$.

The following example shows that there exist GFMRAs which can not derive orthonormal wavelets.

Example 2 Let $\varphi(t)$ satisfy $\widehat{\varphi}(\omega) = I_{[-\frac{3\pi}{8}, \frac{\pi}{8}]}(\omega)$ and $V_m = \overline{\mathrm{span}}\{\varphi(2^m t-n),\ n \in Z\},\ m \in Z$.

A similar argument to that used in Example 1 shows that $\{V_m\}$ is a GFMRA and

$$W_0 = \left\{ f \in L^2(R),\ \mathrm{supp}\widehat{f} \subset \left[-\frac{3\pi}{4}, -\frac{3\pi}{8} \right] \bigcup \left[\frac{\pi}{8}, \frac{\pi}{4} \right] \right\}, \tag{5.4}$$

where W_0 is the orthogonal complement of V_0 in V_1

Suppose that there exists an orthonormal wavelet $\{\psi^{(j)}(t)\}_1^p$ which is derived by this GFMRA $\{V_m\}$. By the definition, we know that $\{\psi^{(j)}(t-n)\}_n$ is an orthonormal system. Again, by Proposition 2(iii), we have

$$[\widehat{\psi}^{(j)}, \widehat{\psi}^{(j)}](\omega) = 1 \quad \text{a.e. } \omega \in R. \tag{5.5}$$

On the other hand, by $\psi^{(j)} \in W_0$ and (5.4), we have

$$\mathrm{supp}\widehat{\psi}^{(j)} \subset \left[-\frac{3\pi}{4}, -\frac{3\pi}{8} \right] \bigcup \left[\frac{\pi}{8}, \frac{\pi}{4} \right].$$

From this, we conclude that

$$[\widehat{\psi}^{(j)}, \widehat{\psi}^{(j)}](\omega) = \sum_k |\widehat{\psi}^{(j)}(\omega + 2k\pi)|^2 = 0, \quad \omega \in \left(-\frac{3\pi}{8}, \frac{\pi}{8} \right).$$

This is contrary to (5.5). Therefore, this GFMRA can not derive an orthonormal wavelet.

Applying Theorem 1, we can also conclude the above result. In fact, $\tau(\omega) = [\widehat{\varphi}, \widehat{\varphi}](\omega) = I_{[-\frac{3\pi}{8}, \frac{\pi}{8}]}(\omega),\ \omega \in [-\pi, \pi]$, and $\tau(\omega + 2n\pi) = \tau(\omega),\ n \in Z$. Furthermore, we can obtain that

$$\tau\left(\frac{\omega}{2} + \pi\right) + \tau\left(\frac{\omega}{2}\right) - \tau(\omega) = \begin{cases} 1, & \omega \in \left(-\frac{3\pi}{4}, -\frac{3\pi}{8} \right) \bigcup \left(\frac{\pi}{8}, \frac{\pi}{4} \right), \\ 0, & \omega \in \left(-\pi, -\frac{3\pi}{4} \right) \bigcup \left(-\frac{3\pi}{8}, \frac{\pi}{8} \right) \bigcup \left(\frac{\pi}{4}, \pi \right). \end{cases}$$

So, by Theorem 1, we know that this GFMRA can not derive an orthonormal wavelet.

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