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A RECONSTRUCTION FORMULA OF THE SIGNAL BASED ON A NEW KIND OF HALF-DISCRETE WAVELET TRANSFORM*

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Abstract On condition that the basic wavelet $\Psi(t)$ is bandimited, a reconstruction formula of the signal $f(t)$ based on a half-discrete wavelet transform $(W_\Psi f)(na,a)(n\in Z,a\in R)$ is given.

Keywords Wavelet transform, basic wavelet, reconstruction, band-limit

1. Introduction and main results

Let $\hat{h}$ be the Fourier transform of h, $\|\cdot\|$ and $(\ ,\ )$ be the norm and the inner product in $L^2(R)$, supp h=clos$\{t\in R,h(t)\neq 0\}$

If $\Psi\in L^2(R)$ and $C_\Psi=2\pi\int_{-\infty}^{\infty}\frac{|\hat{\Psi}(\omega)|^2}{|\omega|}d\omega<\infty$ (1)

the Ψ is said to be a basic wavelet[1].

$$(W_\Psi f)(b,a) = |a|^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(t)\,\overline{\Psi(\frac{t-b}{a})}dt, \tag{2}$$

is said to be a wavelet transform[1]. Its reconstruction formula[1] is stated as follows:

$$f(t) = \lim_{\substack{A_1>0,A_1\to 0\\ A_2\to\infty,B\to\infty}} \frac{1}{C_\Psi}\iint_{\substack{A_1\le|a|\le A_2\\ |b|\le B}} (W_\Psi f)(b,a)\Psi(\frac{t-b}{a})|a|^{-\frac{5}{2}}dadb \qquad (L^2).$$

In this paper, on condition that the $\Psi(t)$ is bandlimited, we reconstruct the signal $f(t)$ only using the values $(W_\Psi f)(na,a)(n\in Z,a\in R)$ which is a new kind of half-discrete wavelet transform, different form the ordinary dyadic avelet transform $2^{\frac{j}{2}}(W_\Psi f)(b,\frac{1}{2^j})$ $(j\in Z,b\in R)$[2][3].

First we prove a corresponding Parseval equality.

Theorem 1. If the basic wavelet $\Psi(t)\in L^2\cap L^1(R)$, supp$\hat{\Psi}\subset[-\pi,\pi]$, then for $f,g\in L^2(R)$, have

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\Psi f)(na,a)\,\overline{(W_\Psi g)(na,a)}\,\frac{da}{|a|} = C_\Psi(f,g).$$

With help of Theorem 1, we give a reconstruction formula as follows.

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Theorem 2. Under the condition of Theorem 1, for $f\in L^2(R)$, have

$$f(t)=\lim_{\substack{A_1>0,A_1\to 0,A_2\to\infty\\ m_1\to-\infty,m_2\to\infty}}\frac{1}{C_\Psi}\sum_{m_1}^{m_2}\int_{A_1\le|a|\le A_2}(W_\Psi f)(na,a)|a|^{-\frac{3}{2}}\Psi\left(\frac{t-na}{a}\right)da \qquad (L^2).$$

2. The proof of Theorem 1

By (2),

$$(W_\Psi f)(na,a)=|a|^{-\frac{1}{2}}\int_{-\infty}^{\infty}f(t)\overline{\Psi\left(\frac{t-na}{a}\right)}dt.$$

Applying Parseval equality in Fourier transform and supp $\hat{\Psi}(\omega)\subset[-\pi,\pi]$, it follows that

$$\begin{aligned}(W_\Psi f)(na,a)&=|a|^{\frac{1}{2}}\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{\Psi}(a\omega)e^{-ian\omega}}d\omega\\&=\mathrm{sgn}a\cdot|a|^{\frac{1}{2}}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\hat{f}(\omega)\overline{\hat{\Psi}(\omega)e^{-ian\omega}}d\omega\\&=\mathrm{sgn}a\cdot|a|^{-\frac{1}{2}}\int_{-\pi}^{\pi}\hat{f}\left(\frac{\omega}{a}\right)\overline{\hat{\Psi}(\omega)e^{-in\omega}}d\omega\end{aligned}$$

From $\Psi\in L^1(R)$, have $\hat{\Psi}\in L^\infty(R)$, so $\hat{f}(\frac{\omega}{a})\overline{\hat{\Psi}(\omega)}\in L^2[-\pi,\pi]$.

Similarly,

$$(W_\Psi g)(na,a)=\mathrm{sgn}a\cdot|a|^{-\frac{1}{2}}\int_{-\pi}^{\pi}\hat{g}\left(\frac{\omega}{a}\right)\overline{\hat{\Psi}(\omega)e^{-in\omega}}d\omega,\ \hat{g}\left(\frac{\omega}{a}\right)\overline{\hat{\Psi}(\omega)}\in L^2[-\pi,\pi].$$

Again applying Parseval equality in Fourier series, we have

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(W_\Psi f)(na,a)\overline{(W_\Psi g)(na,a)}\\&=|a|^{-1}\sum_{n=-\infty}^{\infty}\left(\int_{-\pi}^{\pi}\hat{f}\left(\frac{\omega}{a}\right)\overline{\hat{\Psi}(\omega)e^{-in\omega}}d\omega\right)\overline{\left(\int_{-\pi}^{\pi}\hat{g}\left(\frac{\omega}{a}\right)\overline{\hat{\Psi}(\omega)e^{-in\omega}}d\omega\right)}\\&=2\pi|a|^{-1}\int_{-\pi}^{\pi}\hat{f}\left(\frac{\omega}{a}\right)\overline{\hat{g}\left(\frac{\omega}{a}\right)}|\hat{\Psi}(\omega)|^2d\omega,\end{aligned}$$

so from supp $\hat{\Psi}\subset[-\pi,\pi]$, have,

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}(W_\Psi f)(na,a)\overline{(W_\Psi g)(na,a)}\right)\frac{da}{|a|}\\&=2\pi\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{f}\left(\frac{\omega}{a}\right)\overline{\hat{g}\left(\frac{\omega}{a}\right)}|\hat{\Psi}(\omega)|^2d\omega\right)\frac{da}{|a|^2}\\&=2\pi\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{g}(\omega)}|\hat{\Psi}(a\omega)|^2d\omega\right)\frac{da}{|a|}\end{aligned}\qquad(3)$$

Since $\Psi(t)$ is a basic wavelet, it follows from (1) that

$$2\pi\int_{-\infty}^{\infty}\frac{|\hat{\Psi}(a\omega)|^2}{|a|}da=2\pi\int_{-\infty}^{\infty}\frac{|\hat{\Psi}(a)|^2}{|a|}da=C_\Psi.$$

From this and $\hat{f}\,\overline{\hat{g}}\in L^1(R)$, using Fubini theorem and Parseval equality, we have by (3)

$$\int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}(W_\Psi f)(na,a)\overline{(W_\Psi g)(na,a)}\right)\frac{da}{|a|}=C_\Psi\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega=C_\Psi(f,g)\qquad(4)$$

Specially for $f=g$, have

$$\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|(W_\Psi f)(na,a)|^2\frac{da}{|a|}=C_\Psi\|f\|^2 \tag{5}$$

From(5)and Cauchy inequality, have

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}|(W_\Psi f)(na,a)(W_\Psi g)(na,a)|\frac{da}{|a|}$$

$$\le\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|(W_\Psi f)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|(W_\Psi g)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}$$

$$\le\left(\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}|(W_\Psi f)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}\left(\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}|(W_\Psi g)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}<\infty, \tag{6}$$

so by Lebesgue's theorem on dominate convergence, we know that the order of summation and integration can be interchanged in (4), Theorem 1 is proved.

3. The proof of theorem 2

Let

$$\Delta(M,A)=\|f(t)-J_{M,A}(t)\|,$$

where

$$J_{M,A}(t)=\frac{1}{C_\Psi}\sum_{n=m_1}^{m_2}\int_{A_1\le|a|\le A_2}W_\Psi(na,a)|a|^{-\frac{3}{2}}\Psi\left(\frac{t-na}{a}\right)da.$$

By a known formula, we have

$$\Delta(M,A)=\sup_{\|g\|=1}|(f,g)-(J_{M,A},g)|. \tag{7}$$

By the known conditions and Cauchy inequality, we have

$$|(W_\Psi f)(na,a)|\le\|f\|\cdot\|\Psi\|$$

and

$$\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{|a|}}\Psi\left(\frac{t-na}{a}\right)\overline{g(t)}\right|dt\le\|g\|\cdot\|\Psi\|.$$

From this, using Fubini theorem it follows from(2), that

$$(J_{M,A},g)=\frac{1}{C_\Psi}\sum_{n=m_1}^{m_2}\int_{A_1\le|a|\le A_2}(W_\Psi f)(na,a)\left(\int_{-\infty}^{\infty}\overline{g(t)}\Psi(\frac{t-na}{a})dt\right)\frac{da}{|a|^{\frac{3}{2}}}$$

$$=\frac{1}{C_\Psi}\sum_{n=m_1}^{m_2}\int_{A_1\le|a|\le A_2}(W_\Psi f)(na,a)\,\overline{(W_\Psi g)(na,a)}\,\frac{da}{|a|}. \tag{8}$$

Combining the formula (7) with (8), applying Theorem 1, we finally obtain that

$$\Delta(M,A)=\sup_{\|g\|=1}\{I_1+I_2\}, \tag{9}$$

where

$$I_1=\frac{1}{C_\Psi}\left(\sum_{n=-\infty}^{\infty}-\sum_{n=m_1}^{m_2}\right)\int_{-\infty}^{\infty}(W_\Psi f)(na,a)\,\overline{(W_\Psi g)(na,a)}\,\frac{da}{|a|},$$

$$I_2=\frac{1}{C_\Psi}\sum_{n=m_1}^{m_2}\left(\int_{-\infty}^{\infty}-\int_{A_1\le|a|\le A_2}\right)(W_\Psi f)(na,a)\,\overline{(W_\Psi g)(na,a)}\,\frac{da}{|a|}.$$

But similar to (6), have

$$|I_1|\le\left(\frac{1}{C_\Psi}\left(\sum_{n=-\infty}^{\infty}-\sum_{n=m_1}^{m_2}\right)\int_{-\infty}^{\infty}|(W_\Psi f)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}.$$

$$\left(\frac{1}{C_\Psi}\left(\sum_{n=-\infty}^{\infty}-\sum_{n=m_1}^{m_2}\right)\int_{-\infty}^{\infty}|(W_\Psi g)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}=I_{11}\cdot I_{12}.$$

From (5), we know $\lim_{\substack{m_2\to\infty\\ m_1\to-\infty}} I_{11}=0$ and $I_{12}\leq\|g\|$. So $\sup_{\|g\|=1}|I_1|=0 \quad (m_2\to\infty, m_1\to-\infty)$.

Similarly, from (5),

$$\sup_{\|g\|=1}|I_2|\leq\left(\frac{1}{C_\Psi}\left(\int_{-\infty}^{\infty}-\int_{A_1\leq|a|\leq A_2}\right)\sum_{n=-\infty}^{\infty}|(W_\Psi f)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}.$$

$$\sup_{\|g\|=1}\left(\frac{1}{C_\Psi}\left(\int_{-\infty}^{\infty}-\int_{A_1\leq|a|\leq A_2}\right)\sum_{n=-\infty}^{\infty}|(W_\Psi g)(na,a)|^2\frac{da}{|a|}\right)^{\frac{1}{2}}=0(1)$$

$$(A_1>0, A_1\to 0, A_2\to\infty).$$

From this and (9), we get Theorem 2.

Similar to the proof of Theorem 1 and 2, we have

Corollary Let $\Psi_1(t),\Psi_2(t)\in L^1\cap L^2(R)$, $\mathrm{supp}\,\hat{\Psi}_1\subset[-\pi,\pi]$,

$$\mathrm{supp}\hat{\Psi}_2\subset[-\pi,\pi] \text{ and } \int_{-\infty}^{\infty}\frac{|\hat{\Psi}_1(\omega)\hat{\Psi}_2(\omega)|}{|\omega|}d\omega<\infty.$$

$$C_{\Psi_1,\Psi_2}=2\pi\int_{-\infty}^{\infty}\frac{\overline{\hat{\Psi}_1(\omega)}\hat{\Psi}_2(\omega)}{|\omega|}d\omega.$$

a) For $f,g\in L^2(R)$, we have

$$\int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}(W_{\Psi_1}f)(na,a)\,\overline{(W_{\Psi_2}g)(na,a)}\right)\frac{da}{|a|}=C_{\Psi_1,\Psi_2}(f,g).$$

b) If $C_{\Psi_1,\Psi_2}\neq 0$, then for $f\in L^2(R)$,

$$f(t)=\lim_{\substack{A_1>0,A_1\to 0,A_2\to\infty\\ m_1\to-\infty,m_2\to+\infty}}\frac{1}{C_{\Psi_1,\Psi_2}}\sum_{n=m_1}^{m_2}\int_{A_1\leq|a|\leq A_2}(W_{\Psi_1}f)(na,a)|a|^{-\frac{3}{2}}\Psi_2\left(\frac{t-na}{a}\right)da \qquad (L^2).$$

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基于一类新的半离散的小波变换的信号重构公式

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摘　要　在基小波 $\Psi(t)$ 为带限的条件下，基于半离散的小波变换

$$(W_\Psi f)(na,a) \quad (n\in Z, a\in R)$$

信号 $f(t)$ 的重构公式被给出.

关键词　小波变换，基小波，重构，带限