SHORT COMMUNICATIONS

An improvement of Papadakis' theorem*

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Abstract There exist many orthonormal wavelets which cannot be derived by multiresolution analysis (MRA) with a single scaling function. In 2000, Papadakis announced that any orthonormal wavelet is derived by a generalized MRA with countable scaling functions at most. We improve Papadakis' theorem and find that for any othonormal wavelet, the least number of the corresponding scaling functions is just the essential supremum of the dimension function of the orthonormal wavelet. Moreover, we construct directly the fewest scaling functions.

Keywords: scaling function, generalized MRA, dimension function.

Mallat[1] introduced the classical multiresolution analysis (MRA) with a single scaling function. It is the best approach to the constructions of orthonormal wavelets. However, there indeed exist many orthonomal wavelets which cannot be derived by MRAs with a single scaling function, for example, Journe wavelet[2]. In 2000, Papadakis[3] announced that any orthonormal wavelet is derived by a generalized MRA with countable scaling function at most[3]. This is a very important result in wavelet analysis. As the continuation of Papadakis' work, in this paper, we go deep into the study of the problem of the number of scaling functions. We indicate that the least number of scaling functions is just the essential supremum of the dimension function of the orthonormal wavelet. Meanwhile, we construct directly the fewest scaling functions in the proof of Theorem 1. To delete the redundant part of each scaling function in countable scaling functions, and then reconstruct the fewest scaling functions, this is important in both theory and application. Specially, our construction method in this paper is also new. In order to compare our result with Papadakis' theorem, we take the Journe wavelet for an example.

1 Preliminaries

The definition of generalized MRA and some results in the theory of shift-invariant subspaces are indicated as the following.

Denote the Fourier transform

$$\hat{f}(\omega)=\int_R f(t)\mathrm{e}^{-it\omega}\mathrm{d}t,$$

"span" expresses the linear combination, $I_\Omega(\omega)$ is the characteristic function on Ω, and the notation "$a.e.$" is often omitted.

A subspace U of $L^2(R)$ is said to be shift-invariable if $f(t-n)\in U$ for any $f\in U$ and $n\in Z$.

Let $\{f_l\}_1^r\subset L^2(R)$ and $S=S(\{f_l\}_1^r):=\overline{\text{span}}\{f_l(t-n), n\in Z, l=1,\cdots,r\}$ ($r\in Z^+$ or $r=\infty$). It is clear that S is a shift-invariant subspace of $L^2(R)$. Specially, $S(f)=:\overline{\text{span}}\{f(t-n), n\in Z\}$.

Definition 1. Let $\{V_m\}_{m\in Z}$ be a sequence of closed subspaces of $L^2(R)$ and satisfy

(i) $V_m\subset V_{m+1}$, $m\in Z$, $\overline{\bigcup_m V_m}=L^2(R)$, $\bigcap_m V_m=\{0\}$;

(ii) $f(t)\in V_m\leftrightarrow f(2t)\in V_{m+1}$, $m\in Z$;

(iii) there exists a sequence $\{\varphi^{(j)}\}$ in V_0 such

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that $V_0 = S(\{\varphi^{(j)}\})$.

Then $\{V_m\}_{m\in Z}$ is called a generalized MRA (GMRA) with scaling functions $\{\varphi^{(j)}\}$.

Remark 1. If $\{\varphi(t-n), n\in Z\}$ is an orthonormal basis for V_0, then the GMRA reduces to an MRA.

Remark 2. By the following Lemma 4 and a characterization of frames[4], we see easily that Definition 1 and Papadakis' definition of generalized (frame) MRA[3] are equivalent.

$\psi(t)$ is an orthonormal wavelet if $\{2^{\frac{m}{2}}\psi(2^m t - n), m, n\in Z\}$ is an orthonormal basis for $L^2(R)$.

Let $\{V_m\}$ be a GMRA with many scaling functions, W_0 an orthogonal complement of V_0 in V_1 ($W_0 \oplus V_0 = V_1$) and $\psi \in L^2(R)$ an orthonormal wavelet. If $\{\psi(t-n)\}_{n\in Z}$ is an orthonormal basis for W_0, then we say that ψ is an orthonormal wavelet derived by (or associated with) this GMRA $\{V_m\}$.

Journe showed that there exist orthonormal wavelets which cannot be derived by MRAs with a single scaling function. Later on, in 2000, Papadakis gave the following:

Papadakis' theorem[3]. Any orthonormal wavelet is derived by a GMRA with countable scaling functions at most.

Let $f, g\in L^2(R)$. Denote the bracket product

$$[f,g](\omega) := \sum_{k=-\infty}^{\infty} f(\omega+2k\pi)\bar{g}(\omega+2k\pi) \quad \forall f, g \in L^2(R),$$

and the above series is convergent absolutely and $[f,g](\omega+2\pi) = [f,g](\omega)$ for $a.e.\ \omega\in R$.

Definition 2[2]. Let $\psi\in L^2(R)$. The function $D_\psi(\omega) = \sum_{m<0}[\hat{\psi}(2^{-m}\cdot), \hat{\psi}(2^{-m}\cdot)](\omega)$ is said to be the dimension function of ψ.

We need some results in theory of shift-invariant subspaces, which play a key role in argument of this paper.

Let S be a shift-invariant subspace. If $g\in S$, then the vector $(\hat{g}(\omega+2k\pi))_{k\in Z}\in l^2$ for $a.e.\ \omega\in R$. Denote the corresponding subspace of l^2: $J_S(\omega) = \{(\hat{g}(\omega+2k\pi))_{k\in Z}, g\in S\}$ for fixed $\omega\in R$.

Lemma 1[5]. Let $\{f_l\}_1^r \subset L^2(R)$ and $S = S(\{f_l\}_1^r)$ ($r\in Z^+$ or $r=\infty$). Then $J_S(\omega) = \overline{\text{span}}\{(\hat{f}_l(\omega+2k\pi))_{k\in Z}, l=1,\cdots,r\}$.

Lemma 2[6]. Let S_1, S_2 and S_3 be the shift-invariant subspaces of $L^2(R)$ and $S_1 = S_2\oplus S_3$ ($\oplus$ is the orthogonal sum). Then $J_{S_1}(\omega) = J_{S_2}(\omega)\oplus J_{S_3}(\omega)$ for $a.e.\ \omega\in R$.

Lemma 3[5]. If $f, g\in L^2(R)$, then $S(f)$ and $S(g)$ are orthogonal if and only if $[\hat{f},\hat{g}](\omega) = 0$ $a.e.\ \omega\in R$.

Lemma 4[5]. Suppose that $U = S(\{\varphi^{(j)}\}_1^r)\subset L^2(R)$ ($r\in Z^+$ or $r=\infty$). Then there exists $\{\psi^{(j)}\}_1^r \subset L^2(R)$ such that $U = S(\{\psi\}_1^r)$ and for all j, l,

$$[\hat{\psi}^{(j)},\hat{\psi}^{(l)}](\omega) = \begin{cases} 1 \text{ or } 0, & j=l, \\ 0, & j\neq l, \end{cases} \quad a.e.\ \omega\in R;$$

and for any $f\in U$,

$$\hat{f}(\omega) = \sum_{l=1}^{r}[\hat{f},\hat{\psi}^{(l)}](\omega)\hat{\psi}^{(l)}(\omega)(L^2)$$

and $[\hat{f},\hat{\psi}^{(l)}](\omega)\in L^2_{2\pi}$.

Lemma 5[7]. Let ψ be an orthonormal wavelet. Then the dimension function is

$$D_\psi(\omega) = \dim\{\overline{\text{span}}\{(\hat{\psi}(2^{-m}(\omega+2k\pi)))_{k\in Z}, m<0\}\},$$

where $D_\psi(\omega)$ is stated in Definition 2.

Lemma 6[7]. If ψ is an orthonormal wavelet and the subspace $U = \overline{\text{span}}\{\psi(2^m t - n), m<0, n\in Z\}$, then U is a shift-invariant subspace and $U = S(\{\psi(2^m\cdot)\}_{m<0})$.

2 Main result and its proof

We improve Papadakis' theorem here and show the least number of scaling functions as follows.

Theorem 1. Let ψ be an orthonormal wavelet. Then ψ is derived by a GMRA $\{V_m\}$ with r scaling functions, where $r = \text{ess sup} D_\psi(\omega)$ on R and r is the least number of the scaling functions.

Proof. Using Papadakis' theorem, we know that ψ is derived by a GMRA $\{V_m\}_{m\in Z}$ with the scaling functions $\{\varphi^{(j)}\}_1^\infty$. Further by $V_0 = S(\{\varphi^{(j)}\}_1^\infty)$ and Lemma 4, it follows that there exists a family

$\{\zeta^{(j)}\}_1^\infty \subset L^2(R)$ such that

$$V_0 = S(\{\zeta^{(j)}\}_1^\infty);$$

$$[\hat\zeta^{(j)}, \hat\zeta^{(l)}](\omega) = \begin{cases} 1 \text{ or } 0, & j = l, \\ 0, & j \neq l \end{cases} \quad a.e.\ \omega \in R, \tag{1}$$

and

$$\hat f(\omega) = \sum_{l=1}^{\infty} H_l(\omega)\hat\zeta^{(l)}(\omega) \quad (L^2) \quad \forall f \in V_0,$$

where $$H_l(\omega) \in L^2_{2\pi}. \tag{2}$$

Let W_m be an orthonormal complement of V_m in $V_{m+1}(W_m \oplus V_m = V_{m+1})$. Since $\{\psi(t-n)\}_{n\in Z}$ is an orthonormal basis for W_0 and $V_0 = \bigoplus_{-\infty}^{-1} W_m$, we know that $V_0 = \overline{\text{span}}\ \{\psi(2^m t - n), m < 0, n \in Z\}$. So by Lemma 6, we have $V_0 = S(\{\psi(2^m\cdot)\}_{m<0})$. Again by Lemma 1,

$$J_{V_0}(\omega) = \overline{\text{span}}\ \{(\hat\psi(2^{-m}(\omega + 2k\pi)))_{k\in Z}, m < 0\}$$
$$a.e.\ \omega \in R.$$

Since ψ is an orthonormal wavelet, by Lemma 5, the dimension of the subspace $J_{V_0}(\omega)$ of l^2: $\dim J_{V_0}(\omega) = D_\psi(\omega)$. Further, letting $r = \text{ess sup} D_\psi(\omega)$ on R, we have

$$\text{ess sup}(\dim J_{V_0}(\omega)) = r \text{ on } R. \tag{3}$$

Without loss of generality, we may assume that $r < \infty$. Otherwise, if $r = \infty$, then $\{\varphi^{(j)}\}_1^\infty$ is just the desired r scaling functions. Below we construct the r scaling functions.

Let

$$E_0 := R,$$
$$E_j := \{\omega \in R \mid [\hat\zeta^{(j)}, \hat\zeta^{(j)}](\omega) = 0\}(j \in Z^+);$$
$$F_l := \bigcap_{j=0}^{l} E_j \quad (0 \leqslant l < \infty), \quad F := \bigcap_{j=0}^{\infty} E_j.$$

So from $F_{l-1} \setminus E_l = F_{l-1} \setminus F_l$, we can see that

$$R \setminus F = \bigcap_{l=1}^{\infty} (F_{l-1} \setminus E_l) \quad (\text{disjoint union}). \tag{4}$$

Again define a family of functions $\{g_1^{(j)}\}_1^\infty$ as

$$g_1^{(1)}(\omega) = \sum_{l=1}^{\infty} I_{F_{l-1}\setminus E_l}(\omega)\hat\zeta^{(l)}(\omega);$$
$$g_1^{(j)}(\omega) = I_{R\setminus F_{j-1}}(\omega)\hat\zeta^{(j)}(\omega) \quad (j \geqslant 2). \tag{5}$$

So by (4), we have

$$g_1^{(1)}(\omega) = \begin{cases} \hat\zeta^{(l)}(\omega), & \omega \in F_{l-1}\setminus E_l,\ l = 1,2,\cdots, \\ 0, & \omega \in F. \end{cases}$$
$$g_1^{(j)}(\omega) = \begin{cases} \hat\zeta^{(j)}(\omega), & \omega \in R \setminus F_{j-1}, \\ 0, & \omega \in F_{j-1} \end{cases} \quad (j \geqslant 2). \tag{6}$$

From this and $E_l + 2k\pi = E_l$, $F_l + 2k\pi = F_l$, $F + 2k\pi = F(k \in Z, l \in Z^+)$, it follows that

$$[g_1^{(1)}, g_1^{(1)}](\omega) = \sum_{k=-\infty}^{\infty} \mid g_1^{(1)}(\omega + 2k\pi) \mid^2 = \begin{cases} [\hat\zeta^{(l)}, \hat\zeta^{(l)}](\omega), & \omega \in F_{l-1}\setminus E_l, \\ 0, & \omega \in F \end{cases} \quad (l \in Z^+).$$

Again by the definition of E_l and (1), we have $[\hat\zeta^{(l)}, \hat\zeta^{(l)}](\omega) = 1$, $\omega \in F_{l-1} \setminus E_l$. Further by (4), we obtain immediately that

$$[g_1^{(1)}, g_1^{(1)}](\omega) = 1, \quad \omega \in R \setminus F.$$

And we also obtain that

$$\int_R \mid g_1^{(1)}(\omega) \mid^2 d\omega = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \mid g_1^{(1)}(\omega + 2k\pi) \mid^2 d\omega = \int_{-\pi}^{\pi} [g_1^{(1)}, g_1^{(1)}](\omega) d\omega \leqslant 2\pi.$$

So $g_1^{(1)} \in L^2(R)$. For $j \geqslant 2$, by $\hat\zeta^{(j)} \in L^2(R)$ and (6), we get $g_1^{(j)} \in L^2(R)$ $(j \geqslant 2)$.

Define functions $\{R_1^{(j)}(t)\}_1^\infty$ as

$$\hat R_1^{(j)}(\omega) = g^{(j)}(\omega), \quad j \in Z^+. \tag{7}$$

So we have

$$[\hat R_1^{(1)}, \hat R_1^{(1)}](\omega) = [g_1^{(1)}, g_1^{(1)}](\omega) = 1,$$
$$\omega \in R \setminus F. \tag{8}$$

By (6), (7) and (1), we obtain easily that for j, $l \in Z^+$,

$$[\hat R_1^{(j)}, \hat R_1^{(j)}](\omega) = 0, \quad \omega \in F;$$
$$[\hat R_1^{(j)}, \hat R_1^{(l)}](\omega) = 0 \quad (j \neq l)\ a.e.\ \omega \in R. \tag{9}$$

Now we prove that

$$V_0 = S(\{R_1^{(j)}\}_1^\infty). \tag{10}$$

Since $R = (R \setminus F_{l-1}) \cup (F_{l-1} \setminus E_l) \cup F_l$ (disjoint union),

$$1 = I_{R\setminus F_{l-1}}(\omega) + I_{F_{l-1}\setminus E_l}(\omega) + I_{F_l}(\omega), \quad \omega \in R.$$

By the definition of E_j and $F_l \subset E_l$, we get $\hat\zeta^{(l)}(\omega) = 0$, $\omega \in F_l$. So for any $f \in V_0$, by (2), we have

$$\hat f(\omega) = \sum_{l=1}^{\infty} H_l(\omega) I_{F_{l-1}\setminus F_l}(\omega)\hat\zeta^{(l)}(\omega) + \sum_{l=2}^{\infty} H_l(\omega) I_{R\setminus F_{l-1}}(\omega)\hat\zeta^{(l)}(\omega) \quad (L^2).$$

Again by (6) and (7),

$$\hat f(\omega) = \sum_{l=1}^{\infty} H_l^*(\omega)\hat R_1^{(l)}(\omega) \quad (L^2), \tag{11}$$

where

$$H_1^*(\omega) = \sum_{l=1}^{\infty} H_l(\omega) I_{F_{l-1} \setminus F_l}(\omega) \quad (L_{2\pi}^2),$$
$$H_l^*(\omega) = H_l(\omega) I_{R \setminus F_{l-1}}(\omega) \quad (l \geqslant 2).$$

Since $F_{l-1} \setminus E_l + 2k\pi = F_{l-1} \setminus E_l$, $R \setminus F_{l-1} + 2k\pi = R \setminus F_{l-1} (k \in Z)$, we know that $H_l^*(\omega) \in L_{2\pi}^2$. Letting $H_l^*(\omega) = \sum_{n=-\infty}^{\infty} \beta_n^{(l)} e^{-in\omega}$ and then taking the inverse Fourier transform in (11), we can get

$$f(t) = \sum_{l=1}^{\infty} \Big(\sum_{n=-\infty}^{\infty} \beta_n^{(l)} R_1^{(l)}(t-n) \Big) \quad (L^2).$$

So $f \in S(\{R_1^{(j)}\}_1^{\infty})$. Further, $V_0 \subset S(\{R_1^{(j)}\}_1^{\infty})$.

On the other hand, by (5) and (7), we obtain that $\hat{R}_1^{(1)}(\omega) = \sum_{l=1}^{\infty} I_{F_{l-1 \setminus E_l}}(\omega) \hat{\zeta}^{(l)}(\omega)$ $a.e.$ $\omega \in R$. and this formula holds in L^2. From $I_{F_{l-1} \setminus E_l}(\omega) \in L_{2\pi}^2$ and the first formula in (1), it follows easily that $R_1^{(1)} \in V_0$. Similarly, we have $R_1^{(j)} \in V_0 (j \geqslant 2)$. Further $S(\{R_1^{(j)}\}_1^{\infty}) \subset V_0$. So (10) holds.

By (10) and Lemma 1, we have $J_{V_0}(\omega) = \overline{\text{span}} \{(\hat{R}_1^{(j)}(\omega + 2k\pi))_k, j \in Z^+\}$. However, by the first formula in (9), we get

$$\hat{R}_1^{(j)}(\omega + 2k\pi) = 0$$
$$a.e.\ \omega \in F\ (j \in Z^+, k \in Z). \tag{12}$$

So $\dim J_{V_0}(\omega) = 0$, $\omega \in F$. Further by (3),

$$\text{ess sup}(\dim J_{V_0}(\omega)) = r \text{ on } R \setminus F. \tag{13}$$

Denote $X_1 = S(R_1^{(1)})$, $X_1^* = S(\{R_1^{(j)}\}_2^{\infty})$. By (9) and Lemma 3, we know that for $j \geqslant 2$, the spaces $S(R_1^{(1)})$ and $S(R_1^{(j)})$ are orthogonal. So by (10), we obtain that $V_0 = X_1^* \oplus X_1$. Further, by Lemma 2, $J_{V_0}(\omega) = J_{X_1}(\omega) \oplus J_{X_1^*}(\omega)$. Noticing that $X_1 = S(R_1^{(1)})$, by Lemma 1 we have

$$J_{X_1}(\omega) = \text{span} \{(\hat{R}_1^{(1)}(\omega + 2k\pi))_{k \in Z}\}$$
$$a.e.\ \omega \in R.$$

However by (8), we see that the norm of the vector $(\hat{R}_1^{(1)}(\omega + 2k\pi))_{k \in Z}$ is equal to 1 for $a.e.$ $\omega \in R \setminus F$. So $\dim J_{X_1}(\omega) = 1$, $\omega \in R \setminus F$. So by (13), we obtain that

$$\text{ess sup}(\dim J_{X_1^*}(\omega)) = r - 1 \text{ on } R \setminus F. \tag{14}$$

Noticing that $X_1^* = S(\{R_1^{(j)}\}_2^{\infty})$, and by Lemma 1, we get $J_{X_1^*}(\omega) = \overline{\text{span}} \{(\hat{R}_1^{(j)}(\omega + 2k\pi))_{k \in Z}, j \geqslant 2\}$. Again by (12), $\dim J_{X_1^*}(\omega) = 0$ $a.e.$ $\omega \in F$. From this and (14), we obtain finally that

$$\text{ess sup}(\dim J_{X_1^*}(\omega)) = r - 1 \text{ on } R.$$

Denote $h^{(1)}(t) = R_1^{(1)}(t)$.

Now based on the subspace X_1^* and the functions $\{R_1^{(j)}\}_2^{\infty}$, using a similar procedure, we get two subspaces X_2, X_2^* and new functions $\{R_2^{(j)}\}_1^{\infty}$ such that $X_2 = S(R_2^{(1)})$, $X_2^* = S(\{R_2^{(j)}\}_2^{\infty})$, $X_1^* = X_2^* \oplus X_2$, and $\text{ess sup}(\dim J_{X_2^*}(\omega)) = r - 2$ on R. Denote $h^{(2)}(t) = R_2^{(1)}(t)$.

Keeping on doing this procedure again and again, we get finally r functions $\{h^{(j)}\}_1^r$ and subspaces $X_1, \cdots, X_r, X_r^*$ such that $V_0 = X_r^* \oplus \bigoplus_{j=1}^{r} X_j$; $X_j = S(h^{(j)})$ $(j = 1, \cdots, r)$ and $\text{ess sup}(\dim J_{X_r^*}(\omega)) = 0$ on R. So we see that $X_r^* = \{0\}$. Further, we have $V_0 = S(\{h^{(j)}\}_1^r)$. By Definition 1, $\{h^{(j)}\}_1^r$ are r scaling functions of the GMRA $\{V_m\}$ and $r = \text{ess sup} D_\psi(\omega)$.

Now we prove that $\{h^{(j)}\}_1^r$ are the fewest scaling functions. Suppose that s functions $\{\eta^{(j)}\}_1^s$ are also the scaling functions of the same GMRA $\{V_m\}$. So $V_0 = S(\{\eta^{(j)}\}_1^s)$. By Lemma 1, we have $J_{V_0}(\omega) = \text{span} \{(\hat{\eta}^{(j)}(\omega + 2k\pi))_k \quad j = 1, \cdots, s\}$. Clearly, $\dim J_{V_0}(\omega) \leqslant s$, $a.e.$ $\omega \in R$. Again by (3), we have $r \leqslant s$. The proof of Theorem 1 is completed.

As an application of Theorem 1, we discuss the well-known Journe wavelet.

Example. Journe wavelet ψ[2], which is defined as

$$\hat{\psi}(\omega) = I_\Omega(\omega),$$

where
$$\Omega = \Big\{\omega \in R \mid \frac{4\pi}{7} \leqslant |\omega| \leqslant \pi \text{ or } 4\pi \leqslant |\omega| \leqslant \frac{32\pi}{7}\Big\},$$

is an orthonormal wavelet and cannot be derived by an MRA with a single function. Papadakis showed that it is derived by a GMRA with three scaling functions. However, the number "3" is not the least number of scaling functions. On the other hand, we see easily that $\text{ess sup} D_\psi(\omega) = 2$ on R. Again by Theorem 1, we obtain that the least number of scaling functions is 2. So Journe wavelet ψ can be derived by a GMRA with two scaling functions.

Now, let φ_1, φ_2 satisfy $\hat{\varphi}_1(\omega) = I_{\Omega_1}(\omega)$ and $\hat{\varphi}_2(\omega) = I_{\Omega_2}(\omega)$, where

$$\Omega_1 = \left[-\frac{16\pi}{7}, -2\pi\right] \cup \left[-\frac{8\pi}{7}, -\pi\right] \cup \left[-\frac{4\pi}{7}, -\frac{2\pi}{7}\right] \cup \left[0, \frac{2\pi}{7}\right],$$

$$\Omega_2 = -\Omega_1.$$

Again letting $V_0 = S(\{\varphi_1, \varphi_2\})$, $V_m = \{f \in L^2(R) \mid f(2^{-m}\cdot) \in V_0\}$, $m \in Z$, we prove easily that $\{V_m\}$ is a GMRA with two scaling functions φ_1, φ_2 and Journe wavelet ψ is derived by this GMRA.

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