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On a New Kind of Inversion Formula For the Wavelet Transform

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Abstract A new kind of inversion formula of the wavelet transform for band-limited function is given. This formula possesses the more explicit express than the well-known result and it contains a factor that can be chosen freely.

Key words wavelet transform; inversion formula; band-limited functions

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1 Introduction and main result

By $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in the spaces $L^1(R)$ and $L^2(R)$ respectively, and by $(\cdot,\cdot)$ denote the inner product in the space $L^2(R)$.

By $\hat{f}$ and $f^{\vee}$ denote the Fourier transform and the inverse Fourier transform of f respectively.

The convolution of $f(t)$ and $g(t)$ is defined as $(f * g)(t) = \int_R f(t-x)g(x)\mathrm{d}x$.

For $f(t)\in L^2(R)$, denote $f_{b,a}(t)=\frac{1}{\sqrt{|a|}}f\left(\frac{t-b}{a}\right)$ and $\mathrm{supp}f=\mathrm{clos}\{t\in R: f(t)\neq 0\}$. If $\mathrm{supp}\hat{f}$ is a bounded set, then we say f is band-limited.

The characteristic function on a set E is denoted by $X_E(t)$.

In 1984, Morlet introduced first wavelet transform that is defined as follows:

Let $\psi\in L^2(R)$. The transform:

$$(W_\psi f)(b,a) = \int_R f(t)\overline{\psi_{b,a}(t)}\mathrm{d}t \quad \text{for any} \quad f\in L^2(R) \tag{1}$$

is said to be a wavelet transform[1].

When $\psi\in L^1\cap L^2(R)$ and $C_\psi=2\pi\int_R \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\mathrm{d}\omega<\infty$, the known inversion formula is stated as follows[1,2]:

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$$f(t)=\lim_{\substack{A_1\to 0+,\ A_2\to+\infty\\ B\to+\infty}}\frac{1}{C_\psi}\iint_{\substack{A_1\le|a|\le A_2\\ |b|\le B}}(W_\psi f)(b,a)\psi_{b,a}(t)\frac{\mathrm{d}a\mathrm{d}b}{a^2}\quad (L^2). \tag{2}$$

The above equality holds in L^2-sense.

The aim of this paper is that for band-limited function we give another kind of inversion formula of wavelet transform.

Theorem Let $\psi(t)\in L^1\cap L^2(R)$. Take $\varphi(t)\in L^1\cap L^2(R)$ satisfying $\hat\varphi(\omega)=O(|\omega|^{-2})$. Then for any $f\in L^1\cap L^2(R)$ and $\mathrm{supp}\hat f\subseteq[-\Omega,\Omega]$, the following inversion formula holds:

$$f(t)=\lim_{A\to+\infty}\frac{1}{(2\pi)^{\frac{3}{2}}(\varphi,\psi)}\iint_{\substack{|a|\le A\\ |b|<\infty}}(W_\psi f)(b,a)(\varphi_{b,a}*h)(t)\frac{\mathrm{d}a\mathrm{d}b}{|a|}\quad (L^2) \tag{3}$$

where $h(t)$ satisfies $\hat h(\omega)=|\omega|X_{[-\Omega,\Omega]}(\omega)$ and the above equality holds in L^2-sense.

The limit process in (3) is simpler than that in (2), and the function $\varphi(t)$ in (3) can be chosen freely.

2 Lemma

To prove Theorem, we first give the following

Lemma Let $\psi(t)$, $\varphi(t)$ and $f(t)$ be stated in Theorem. Then for any $g\in L^2(R)$, the following formula is valid:

$$\frac{1}{2\pi}\int_{R^2}(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}\frac{\mathrm{d}a\mathrm{d}b}{|a|}=(\varphi,\psi)(f,g),$$

where $(D_\varphi g)(b,a)=\dfrac{1}{\sqrt{2\pi}}(g,\varphi_{b,a}*h)$. (4)

Proof By Parseval equality of Fourier transform and (1), we have

$$(W_\psi f)(b,a)=|a|^{\frac{1}{2}}\int_R\hat f(\omega)\overline{\hat\psi(a\omega)}e^{ib\omega}\mathrm{d}\omega. \tag{5}$$

Using the convolution formula and Parseval equality, we also obtain from (4) that

$$(D_\varphi g)(b,a)=|a|^{\frac{1}{2}}\int_R\hat g(\omega)\overline{\hat\varphi(a\omega)\hat h(\omega)}e^{ib\omega}\mathrm{d}\omega. \tag{6}$$

Applying the inversion formula of Fourier transform, it follows from (5) and (6) that

$$\frac{1}{\sqrt{2\pi}|a|^{\frac{1}{2}}}(W_\psi f)(b,a)=(\hat f(\omega)\overline{\hat\psi(a\omega)})^\vee(b) \tag{7}$$

and

$$\frac{1}{\sqrt{2\pi}|a|^{\frac{1}{2}}}(D_\varphi f)(b,a)=(\hat g(\omega)\overline{\hat\varphi(a\omega)\hat h(\omega)})^\vee(b). \tag{8}$$

Finally again using Parseval equality, we get

$$\frac{1}{2\pi|a|}\int_R(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}\mathrm{d}b=\int_R\hat f(\omega)\overline{\hat g(\omega)}\hat h(\omega)\overline{\hat\psi(a\omega)}\hat\varphi(a\omega)\mathrm{d}\omega.$$

Since $\mathrm{supp}\hat f\subseteq[-\Omega,\Omega]=\mathrm{supp}\hat h(\omega)$ and $\hat h(\omega)=|\omega|X_{[-\Omega,\Omega]}(\omega)$, we know that

$$\hat f(\omega)\hat h(\omega)=|\omega|\hat f(\omega),\quad \omega\in R.$$

Further, $\frac{1}{2\pi|a|}\int_R (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}db = \int_R \hat{f}(\omega)\overline{\hat{g}(\omega)}|\omega|\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)d\omega.$

In view of

$$\int_R\int_R |\hat{f}(\omega)\overline{\hat{g}(\omega)}\omega\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)|d\omega da = \int_R |\hat{f}(\omega)\overline{\hat{g}(\omega)}|\left(|\omega|\int_R|\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)|da\right)d\omega$$

$$= \left(\int_R|\overline{\hat{\psi}(\omega)}\hat{\varphi}(\omega)|d\omega\right)\left(\int_R|\hat{f}(\omega)\overline{\hat{g}(\omega)}|d\omega\right) \le \|\varphi\|_2\|\psi\|_2\|f\|_2\|g\|_2,$$

by Fubini theorem, we have

$$\frac{1}{2\pi}\int_R\left(\int_R(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}db\right)\frac{da}{|a|} = \int_R\left(\int_R\hat{f}(\omega)\overline{\hat{g}(\omega)}|\omega|\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)d\omega\right)da$$

$$= \int_R\hat{f}(\omega)\overline{\hat{g}(\omega)}\left(|\omega|\int_R\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)da\right)d\omega.$$

Again noticing that

$$|\omega|\int_R\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)da = (\hat{\varphi},\hat{\psi}) = (\varphi,\psi),$$

for the repeated integral, we get

$$\frac{1}{2\pi}\int_R\left(\int_R(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}db\right)\frac{da}{|a|} = (\varphi,\psi)(f,g).$$

In order to complete the proof of Lemma, by Fubini theorem, we only need to prove that

$$K = \int_{R^2}|(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}|\frac{dadb}{|a|} < \infty. \tag{9}$$

We split the above integral into two parts, namely,

$$K = \left(\int_{-1}^{1} + \int_{R-[-1,1]}\right)\left(\int_R|(W_\psi f)(b,a)\overline{D_\varphi g)(b,a)}|\frac{db}{|a|}\right)da = K_1 + K_2. \tag{10}$$

First we estimate K_1.

Using Cauchy inequality, we get

$$K_1^2 \le \int_{-1}^{1}\left(\int_R|(W_\psi f)(b,a)|^2\frac{db}{|a|}\right)da \cdot \int_{-1}^{1}\left(\int_R|(D_\varphi g)(b,a)|^2\frac{db}{|a|}\right)da = K_{11}\cdot K_{12}.$$

Applying (7) and Parseval equality, we have

$$K_{11} = 2\pi\int_{-1}^{1}\left(\int_R|\hat{f}(\omega)|^2|\hat{\psi}(a\omega)|^2d\omega\right)da.$$

By $\psi\in L^1(R)$, we know that there is a $M>0$ such that $|\hat{\psi}(\omega)|\le M$, so $K_{11}\le 4\pi M^2\|f\|_2^2$.

On the other hand, applying (8) and Parseval equality, we also have

$$K_{12} = 2\pi\int_{-1}^{1}\left(\int_R|\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2|\hat{h}(\omega)|^2d\omega\right)da. \tag{11}$$

By $\varphi\in L^1(R)$, we know that there is an $N>0$ such that $|\hat{\varphi}(\omega)|\le N$. Again noticing that $|\hat{h}(\omega)|\le\Omega$, we have $K_{12}\le 4\pi N^2\Omega^2\|g\|_2^2$. So $K_1<\infty$.

Next we estimate K_2.

From (10), we know that for any given $0<\varepsilon<\frac{1}{2}$,

$$K_2=\int_{R-[-1,1]}\left(\int_R|a|^{-1+\frac{\varepsilon}{2}}|(W_\psi f)(b,a)|\cdot|a|^{-\frac{\varepsilon}{2}}|(D_\varphi g)(b,a)|\mathrm{d}b\right)\mathrm{d}a.$$

Using Cauchy inequality, we get

$$K_2^2\le\int_{R-[-1,1]}\left(\int_R|(W_\psi f)(b,a)|^2\frac{\mathrm{d}b}{|a|^{2-\varepsilon}}\right)\mathrm{d}a\cdot\int_{R-[-1,1]}\left(\int_R|(D_\varphi g)(b,a)|^2\frac{\mathrm{d}b}{|a|^{\varepsilon}}\right)\mathrm{d}a=K_{21}\cdot K_{22}.$$

Since

$$|(W_\psi f)(b,a)|=\left|\frac{1}{\sqrt{|a|}}\int_R f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\mathrm{d}t\right|\le\|f\|_2\|\psi\|_2$$

and

$$\frac{1}{\sqrt{|a|}}\int_R|(W_\psi f)(b,a)|\mathrm{d}b\le\iint_{RR}\left|f(t)\psi\left(\frac{t-b}{a}\right)\right|\frac{\mathrm{d}t\mathrm{d}b}{|a|}\le\|f\|_1\|\psi\|_1,\qquad(12)$$

we get

$$K_{21}\le\|f\|_2\|\psi\|_2\int_{R-[-1,1]}\left(\int_R|(W_\psi f)(b,a)|\mathrm{d}b\right)\frac{1}{|a|^{2-\varepsilon}}\mathrm{d}a$$

$$\le\|f\|_2\|\psi\|_2\|f\|_1\|\psi\|_1\int_{R-[-1,1]}\frac{1}{|a|^{\frac{3}{2}-\varepsilon}}\mathrm{d}a=\frac{4}{1-2\varepsilon}\|f\|_2\|\psi\|_2\|f\|_1\|\psi\|_1$$

Similar to the argument of (11), we have

$$K_{22}=2\pi\int_{R-[-1,1]}\left(\int_R|\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2|\hat{h}(\omega)|^2|a|^{1-\varepsilon}\mathrm{d}\omega\right)\mathrm{d}a.$$

Further, by the definition of $h(t)$,

$$K_{22}=2\pi\int_{R-[-1,1]}|a|^{-1-\varepsilon}\left(\int_R|\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2|a\omega|^2\mathrm{d}\omega\right)\mathrm{d}a.$$

From $\hat{\varphi}(\omega)=O(|\omega|^{-2})$, we have $|\hat{\varphi}(a\omega)|^2|a\omega|^2\le M_1$ (M_1 is an absolute constant). Further $K_{22}\le\frac{4\pi M_1}{\varepsilon}\|g\|_2^2$. So $K_2<\infty$. We obtain finally (9). The proof of Lemma is completed.

3 Proof of Theorem

From $|(\varphi_{b,a}*h)(t)|\le\|\varphi_{b,a}\|_2\|h\|_2=\|\varphi\|_2\|h\|_2$ and (12), we have

$$\int_{-A}^{A}\int_R|(W_\psi f)(b,a)(\varphi_{b,a}*h)(t)|\frac{\mathrm{d}b\mathrm{d}a}{|a|}\le\|\varphi\|_2\|h\|_2\int_{-A}^{A}\frac{1}{|a|^{\frac{1}{2}}}\left(\frac{1}{|a|^{\frac{1}{2}}}\int_R|(W_\psi f)(b,a)|\mathrm{d}b\right)\mathrm{d}a$$

$$\le4\sqrt{A}\|\varphi\|_2\|h\|_2\|\psi\|_1\|f\|_1\qquad(13)$$

So for all $t\in R$, we know that $(W_\psi f)(b,a)(\varphi_{b,a}*h)(t)\frac{1}{|a|}\in L^1([-A,A]\times R)$.

Set

$$\triangle_A(t)=\frac{1}{(2\pi)^{\frac{3}{2}}(\varphi,\psi)}\int_{-A}^{A}\int_R(W_\psi f)(b,a)(\varphi_{b,a}*h)(t)\frac{\mathrm{d}b\mathrm{d}a}{|a|}.$$

By the known result in theory of Hilbert space, we know that

$$\| f(t) - \triangle_A(t) \|_2 = \sup_{\|g\|_2=1} |(f, g) - (\triangle_A, g)|. \tag{14}$$

Again by (4), we get

$$(\triangle_A, g) = \frac{1}{(2\pi)^{\frac{3}{2}}(\varphi,\psi)} \int_R \left(\int_{-A}^{A} \int_R (W_\psi f)(b,a)(\varphi_{b,a} * h)(t) \overline{g(t)} \frac{\mathrm{d}b\mathrm{d}a}{|a|} \right) \mathrm{d}t$$

$$= \frac{1}{2\pi(\varphi,\psi)} \int_{-A}^{A} \int_R (W_\psi f)(b,a) \overline{(D_\varphi g)(b,a)} \frac{\mathrm{d}b\mathrm{d}a}{|a|}. \tag{15}$$

The reason for interchanging the order of the above integrals is stated as follows.

By (12) and

$$\int_R |(\varphi_{b,a} * h)(t) \overline{g(t)}| \frac{1}{\sqrt{|a|}} \mathrm{d}t \leq \| \frac{1}{\sqrt{|a|}} (\varphi_{b,a} * h)(t) \|_2 \| g \|_2 \leq \| \varphi \|_1 \| h \|_2 \| g \|_2,$$

we get

$$\int_R \left(\int_{-A}^{A} \int_R |(W_\psi f)(b,a)(\varphi_{b,a} * h)(t) \overline{g(t)}| \frac{\mathrm{d}b\mathrm{d}a}{|a|} \right) \mathrm{d}t$$

$$= \int_{-A}^{A} \left(\int_R |(W_\psi f)(b,a)| \left(\int_R |(\varphi_{b,a} * h)(t) \overline{g(t)}| \frac{1}{\sqrt{|a|}} \mathrm{d}t \right) \frac{\mathrm{d}b}{\sqrt{|a|}} \right) \mathrm{d}a$$

$$\leq 2A \| \varphi \|_1 \| h \|_2 \| g \|_2 \| f \|_1 \| \psi \|_1$$

So the order of integrals in (15) can be interchanged.

Using Lemma and (15),

$$(f, g) - (\triangle_A, g) = \frac{1}{2\pi(\varphi,\psi)} \int_{R-[-A,A]} \int_R (W_\psi f)(b,a) \overline{(D_\varphi g)(b,a)} \frac{\mathrm{d}b\mathrm{d}a}{|a|}.$$

Further we get from (14)

$$\| f(t) - \triangle_A(t) \|_2 \leq \sup_{\|g\|_2=1} \left(\frac{1}{2\pi|(\varphi,\psi)|} \int_{R-[-A,A]} \int_R |(W_\psi f)(b,a) \overline{(D_\varphi g)(b,a)}| \frac{\mathrm{d}b\mathrm{d}a}{|a|} \right)$$

$$= \sup_{\|g\|_2=1} \left(\frac{1}{2\pi|(\varphi,\psi)|} I(A) \right), \tag{16}$$

where $I(A) = \int_{R-[-A,A]} \int_R |(W_\psi f)(b,a) \overline{(D_\varphi g)(b,a)}| \frac{\mathrm{d}b\mathrm{d}a}{|a|}.$

Using Cauchy inequality, we can see that

$$I^2(A) \leq \int_{R-[-A,A]} \left(\int_R |(W_\psi f)(b,a)|^2 \frac{\mathrm{d}b}{|a|^{2-\varepsilon}} \right) \mathrm{d}a \cdot \int_{R-[-A,A]} \left(\int_R |(D_\varphi g)(b,a)|^2 \frac{\mathrm{d}b}{|a|^{\varepsilon}} \right) \mathrm{d}a$$

$$= I_1(A) \cdot I_2(A) \tag{17}$$

Imitating the estimates of K_{21} and K_{22} in Lemma, we can get

$$I_1(A) \leq \frac{4}{1-2\varepsilon} A^{-\frac{1}{2}+\varepsilon} \| f \|_2 \| \psi \|_2 \| f \|_1 \| \psi \|_1;$$

and

$$I_2(A) \leq \frac{4\pi M_1}{\varepsilon} A^{-\varepsilon} \| g \|_2^2.$$

From this and (16), (17), we know that

$$\| f(t) - \triangle_A(t) \|_2 \leq \sup_{\|\varphi\|_2=1} \left(\frac{1}{2\pi |(\varphi,\psi)|} I(A) \right)$$

$$\leq \frac{1}{2\pi |(\varphi,\psi)|} \left(\frac{16\pi M_1}{(1-2\varepsilon)\varepsilon} A^{-\frac{1}{2}} \| f \|_2 \| \psi \|_2 \| f \|_1 \| \psi \|_1 \right)^{\frac{1}{2}}.$$

So $\lim\limits_{A \to +\infty} \| f(t) - \triangle_A(t) \|_2 = 0$. The proof of Theorem is completed.

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小波变换的一类新的反演公式

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摘 要 给出关于带限函数的小波变换的一类新的反演公式，这个公式具有比熟知的结果更清晰的表达式，并且含有可以自由选择的因子.

关键词 小波变换；反演公式；带限函数