

§ 2.7

#3 $\vec{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{X} \quad \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = (\lambda+1)(\lambda-1) \quad \lambda = \pm 1$

$\lambda = 1 \quad \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow$ pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solutions

$\vec{X}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda = -1 \quad \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow$ pick $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

compute $\vec{X}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow c_1 = \frac{3}{2} \quad c_2 = -\frac{1}{2}$

compute $\vec{X}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$

→ For $\vec{X}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X(t) = \begin{bmatrix} \frac{3}{2} e^t - \frac{1}{2} e^{-t} \\ \frac{3}{2} e^t - \frac{3}{2} e^{-t} \end{bmatrix}$

$\therefore \Phi(t)$

→ For $\vec{X}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X(t) = \begin{bmatrix} -\frac{1}{2} e^t + \frac{1}{2} e^{-t} \\ -\frac{1}{2} e^t + \frac{3}{2} e^{-t} \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}$

#11 $\vec{X}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{X}_1(0) - 1\vec{X}_2(0) \quad \therefore \vec{X}(t) = \Phi(t) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\therefore \vec{X}(t) = 2 \cdot \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} \\ 3e^t - 3e^{-t} \end{bmatrix} - 1 \begin{bmatrix} -e^t + e^{-t} \\ -e^t + 3e^{-t} \end{bmatrix}$

$= \begin{bmatrix} 3e^t - e^{-t} - (-e^t + e^{-t}) \\ 3e^t - 3e^{-t} - (-e^t + 3e^{-t}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t - \frac{3}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

#14 $\Phi(t) = \frac{1}{4} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix}$

check $\Phi(t)\Phi(s) = \Phi(t+s)$ by yourself.

$$\mathbf{I} + \sum_{k=1}^m \mathbf{A}^k \frac{t^k}{k!} = \begin{pmatrix} \sum_{k=0}^m a_1^k \frac{t^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^m a_2^k \frac{t^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^m a_n^k \frac{t^k}{k!} \end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t . Taking the limit (as $m \rightarrow \infty$) on both sides of the equation, we obtain

$$\exp(\mathbf{A}t) = \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_n t} \end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Since ODEs are *uncoupled*, the vectors $\mathbf{x}^{(j)} = \exp(a_j t) \mathbf{e}^{(j)}$, $j = 1, 2, \dots, n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{X} = [\exp(a_1 t) \mathbf{e}^{(1)}, \exp(a_2 t) \mathbf{e}^{(2)}, \dots, \exp(a_n t) \mathbf{e}^{(n)}]$$

is a *fundamental matrix*. Finally, since $\mathbf{X}(0) = \mathbf{I}$, it follows that

$$[\exp(a_1 t) \mathbf{e}^{(1)}, \exp(a_2 t) \mathbf{e}^{(2)}, \dots, \exp(a_n t) \mathbf{e}^{(n)}] = \mathbf{\Phi}(t) = \exp(\mathbf{A}t).$$

17. (a) Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + \omega^2 x_1 = 0$$

with the initial conditions $x_1(0) = u_0$ and $x_2(0) = v_0$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1 \end{aligned}$$

which can be expressed in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

(b) Setting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

it is easy to show that

$$\mathbf{A}^2 = -\omega^2 \mathbf{I}, \mathbf{A}^3 = -\omega^2 \mathbf{A} \text{ and } \mathbf{A}^4 = \omega^4 \mathbf{I}.$$

It follows inductively that

$$\mathbf{A}^{2k} = (-1)^k \omega^{2k} \mathbf{I}$$

and

$$\mathbf{A}^{2k+1} = (-1)^k \omega^{2k} \mathbf{A}.$$

Hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \sum_{k=0}^{\infty} \left[(-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \mathbf{I} + (-1)^k \frac{\omega^{2k} t^{2k+1}}{(2k+1)!} \mathbf{A} \right] \\ &= \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \right] \mathbf{I} + \frac{1}{\omega} \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k+1} t^{2k+1}}{(2k+1)!} \right] \mathbf{A} \end{aligned}$$

and therefore

$$\exp(\mathbf{A}t) = \cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A}.$$

(c) From Eqn. (28),

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \left[\cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A} \right] \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ &= \cos \omega t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \frac{1}{\omega} \sin \omega t \begin{pmatrix} v_0 \\ -\omega^2 u_0 \end{pmatrix}. \end{aligned}$$

18(a). Assuming that $\mathbf{x} = \phi(t)$ is a solution, then $\phi' = \mathbf{A}\phi$, with $\phi(0) = \mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t) - \phi(0) = \int_0^t \mathbf{A}\phi(s) ds.$$

Hence

$$\phi(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi(s) ds.$$

(b) Proceed with the iteration

$$\phi^{(i+1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi^{(i)}(s) ds.$$

With $\phi^{(0)}(t) = \mathbf{x}^0$, and noting that \mathbf{A} is a constant matrix,

$$\begin{aligned}\phi^{(1)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t.\end{aligned}$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c) We then have

$$\begin{aligned}\phi^{(2)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}s)\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2 \mathbf{x}^0 \frac{t^2}{2} \\ &= \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} \right) \mathbf{x}^0.\end{aligned}$$

Now suppose that

$$\phi^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^n \frac{t^n}{n!} \right) \mathbf{x}^0.$$

It follows that

$$\begin{aligned}\int_0^t \mathbf{A} \left(\mathbf{I} + \mathbf{A}s + \mathbf{A}^2 \frac{s^2}{2} + \cdots + \mathbf{A}^n \frac{s^n}{n!} \right) \mathbf{x}^0 ds &= \\ = \mathbf{A} \left(\mathbf{I}t + \mathbf{A} \frac{t^2}{2} + \mathbf{A}^2 \frac{t^3}{3!} + \cdots + \mathbf{A}^n \frac{t^{n+1}}{(n+1)!} \right) \mathbf{x}^0 &= \\ = \left(\mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1} \frac{t^{n+1}}{(n+1)!} \right) \mathbf{x}^0.\end{aligned}$$

Therefore

$$\phi^{(n+1)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^{n+1} \frac{t^{n+1}}{(n+1)!} \right) \mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d) Define $\phi^{(\infty)}(t) = \lim_{n \rightarrow \infty} \phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = \exp(\mathbf{A}t)\mathbf{x}^0.$$

Term-by-term differentiation results in

$$\begin{aligned}\frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots\right)\mathbf{x}^0 \\ &= \left(\mathbf{A} + \mathbf{A}^2t + \cdots + \mathbf{A}^n\frac{t^{n-1}}{(n-1)!} + \cdots\right)\mathbf{x}^0 \\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n-1}\frac{t^{n-1}}{(n-1)!} + \cdots\right)\mathbf{x}^0.\end{aligned}$$

That is,

$$\frac{d}{dt}\phi^{(\infty)}(t) = \mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0) = \mathbf{x}^0$. Based on *uniqueness* of solutions, $\phi(t) = \phi^{(\infty)}(t)$.

§ 7.8

#1

$$\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x} \quad \left| \begin{array}{cc} 3-\lambda & -4 \\ 1 & -1-\lambda \end{array} \right| = \lambda^2 - 2\lambda - 3 + 4 = \lambda^2 - 2\lambda + 1 = 0$$

$\lambda = 1$ (repeat)

$$\lambda = 1 \quad \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \quad \text{pick} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{pick} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right)$$

#6

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x} \quad \left| \begin{array}{ccc} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{array} \right| = -\lambda^3 + 2 + 3\lambda = -(\lambda+1)^2(\lambda-2) = 0$$

$\lambda = 2, -1, -1,$

$$\lambda = 2 \quad \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{linear independent}$$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t}$$

8

$$\vec{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \vec{x} \quad x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{vmatrix} -\frac{5}{2} - \lambda & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - \lambda \end{vmatrix} = \lambda^2 + 2\lambda - \frac{5}{4} + \frac{9}{4} = \lambda^2 + 2\lambda + 1 = 0 \quad \lambda = -1, -1$$

$$\lambda = -1, \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \quad \text{rank} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{rank} \Rightarrow \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right)$$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left(0 + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \right) \Rightarrow c_1 = 3, \quad c_2 = -6$$

$$\therefore \vec{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + (-6) \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right)$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} -6 \\ -6 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} e^{-t}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t}$$

#17

$$\vec{x}' = Ax = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} x, \quad \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda) + 3 + 4 - 2(4-\lambda) + 3(1-\lambda) + 2(1-\lambda)$$

$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = (\lambda+2)^3 = 0$$

(a)

$$\Rightarrow \lambda = 2, 2, 2.$$

(b)

rank 2 matrix $\dim(\ker) = 3-2=1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0, \quad \text{pick } \xi = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad X^{(1)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$$

(c)

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{pick } \eta = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X^{(2)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

(d)

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{pick } \zeta = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad X^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t e^{2t} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} e^{2t}$$

(e)

$$\Psi(t) = \begin{bmatrix} | & | & | \\ X^{(1)}(t) & X^{(2)}(t) & X^{(3)}(t) \\ | & | & | \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 2+t \\ 1 & 1+t & t+\frac{t^2}{2} \\ -1 & -t & 3-\frac{t^2}{2} \end{bmatrix}$$

(f)

$$T = \begin{bmatrix} | & | & | \\ \xi & \eta & \zeta \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}, \quad \det(T) = 0 - 3 + 2 = -1$$

$$\text{Aug}(T) = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 2 & -1 \\ -2 & +2 & -1 \end{bmatrix}^T, \quad T^{-1} = \frac{\text{Adj}(T)}{\det(T)} = -1 \begin{bmatrix} 3 & -3 & -2 \\ -3 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\text{Jordan form } J = \begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}$$

#19 $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

(a) $J^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}, \quad J^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}, \quad J^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{bmatrix}$ by direct computing!

(b) by math induction, omitted

(c) $Jt = \begin{bmatrix} \lambda t & t \\ 0 & \lambda t \end{bmatrix} \quad e^{Jt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$$e^{Jt} = I + Jt + \frac{1}{2!}(Jt)^2 + \frac{1}{3!}(Jt)^3 + \dots$$

$$= \begin{bmatrix} 1 + t\lambda + \frac{(t\lambda)^2}{2} + \frac{(t\lambda)^3}{3!} + \dots & 0 + t + \frac{2t\lambda}{2!} + \frac{3(t\lambda)^2}{3!} + \dots \\ 0 & 1 + t\lambda + \frac{(t\lambda)^2}{2!} + \frac{(t\lambda)^3}{3!} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

(d) $x' = Jx \quad x(0) = x^0$

$$\Rightarrow x = e^{Jt} x^0 = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} x^0$$

§ 7.9

#1

$$\vec{x}' = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}}_A \vec{x} + \underbrace{\begin{bmatrix} e^t \\ t \end{bmatrix}}_{g(t)}$$

Find the homogeneous solution first. $\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = 0 \Rightarrow \lambda = 1, -1$

$$\lambda = 1, \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = 0 \Rightarrow \zeta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda = -1, \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = 0 \Rightarrow \zeta = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Note that $g(t) = \begin{bmatrix} e^t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$ and $e^t \Leftrightarrow \lambda = 1$,

$$\text{let } \vec{x}(t) = \vec{a} t e^t + \vec{b} e^t + \vec{c} t + \vec{d}$$

$$\vec{x}'(t) = \vec{a} (t e^t + e^t) + \vec{b} e^t + \vec{c} = \vec{a} t e^t + (\vec{a} + \vec{b}) e^t + \vec{c} t + \vec{c}$$

$$\begin{aligned} A \vec{x}(t) + g(t) &= A \vec{a} t e^t + A \vec{b} e^t + A \vec{c} t + A \vec{d} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \\ &= A \vec{a} t e^t + (A \vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}) e^t + (A \vec{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}) t + \vec{d} \end{aligned}$$

$$\therefore \begin{cases} A \vec{a} = \vec{a} & (1) \\ \vec{a} + \vec{b} = A \vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} & (2) \\ A \vec{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 & (3) \\ A \vec{d} = \vec{c} & (4) \end{cases} \quad (3): \begin{cases} 2c_1 - c_2 = 0 \Rightarrow c_1 = 1 \\ 3c_1 - 2c_2 = -1 \Rightarrow c_2 = 2 \end{cases}$$

$$\therefore \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = A \vec{d} \Rightarrow \vec{d} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$(1) \Rightarrow (A - I) \vec{a} = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \vec{a} = a \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(2) \Rightarrow a \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - b_2 \\ 3b_1 - 2b_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} b_1 - b_2 = a - 1 \Rightarrow 2b_1 - 2b_2 = 1 \Rightarrow 2b_1 = 1 + 2b_2 \text{ let } b_2 = b \\ 3b_1 - 3b_2 = a \Rightarrow \vec{b} = \begin{bmatrix} 1 + 2b \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for all } b \end{cases}$$

$$\Rightarrow a = \frac{3}{2}$$

$$\Rightarrow \vec{a} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the simplest choice is $b = 0$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore \vec{x}(t) = \underbrace{C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^t}_{\text{homogeneous sol}} + \underbrace{\frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\text{particular sol}}$$

the constant C_1 adjust the e^t term

#3

$$\vec{X}' = \overbrace{\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}}^A \vec{X} + \overbrace{\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}}^{g(t)}, \quad \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 5 = \lambda^2 + 1 = 0$$

$$\lambda = \pm i \rightarrow e^{i\omega t} = \cos \omega t + i \sin \omega t$$

$$\lambda = i \begin{vmatrix} 2-i & -5 \\ 1 & -2-i \end{vmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0, \quad \vec{z}_1 = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\frac{1}{5}(\cos t + i \sin t) = \begin{bmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{bmatrix} + i \begin{bmatrix} 5 \sin t \\ 2 \sin t - \cos t \end{bmatrix}$$

Note: let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then

$$M^{-1} = \frac{1}{\det M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 5 \cos t & 5 \sin t \\ 2 \cos t + \sin t & 2 \sin t - \cos t \end{bmatrix},$$

$$\det(\Psi) = 10 \cos t \sin t - 5 \cos^2 t - 5 \sin^2 t - 10 \sin t \cos t = -5$$

$$\Psi^{-1} = \frac{1}{-5} \begin{bmatrix} 2 \sin t - \cos t & -(5 \sin t) \\ -(2 \cos t + \sin t) & 5 \cos t \end{bmatrix} = \begin{bmatrix} \frac{\cos t - 2 \sin t}{5} & \sin t \\ \frac{2 \cos t + \sin t}{5} & -\cos t \end{bmatrix} \frac{1}{5}$$

$$\vec{u}'(t) = \Psi^{-1} g = \frac{1}{5} \begin{bmatrix} -\cos^2 t + 2 \sin t \cos t + 5 \sin^2 t \\ -2 \cos^2 t - \cos t \sin t - 5 \cos t \sin t \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} \sin 2t - 1 + 6 \frac{1 - \cos 2t}{2} \\ -2 \frac{1 + \cos 2t}{2} - 6 \sin t \cos t \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 - 3 \cos 2t + \sin 2t \\ -1 - \cos 2t - 3 \sin 2t \end{bmatrix}$$

$$\int 2 - 3 \cos 2t + \sin 2t dt = 2t - \frac{3}{2} \sin 2t - \frac{1}{2} \cos 2t + C_1$$

$$\int -1 - \cos 2t - 3 \sin 2t dt = -t - \frac{1}{2} \sin 2t + \frac{3}{2} \cos 2t + C_2$$

$$\therefore \vec{u}(t) = \underbrace{\begin{bmatrix} \frac{2}{5}t - \frac{3}{10} \sin 2t - \frac{1}{10} \cos 2t \\ -\frac{1}{5}t - \frac{1}{10} \sin 2t + \frac{3}{20} \cos 2t \end{bmatrix}}_{V(t)} + \vec{C} \quad \text{and}$$

Particular sol: $\vec{X}_p(t) = \Psi(t) V(t)$

General sol: $\vec{X}(t) = \underbrace{\Psi(t)}_{\vec{X}_p(t)} \underbrace{u(t)}_{\vec{X}_h(t)} = \Psi(t) V(t) + C_1 \begin{bmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{bmatrix} + C_2 \begin{bmatrix} 5 \sin t \\ 2 \sin t - \cos t \end{bmatrix}$

#11 $\vec{x}' = \overbrace{\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}}^A \vec{x} + \overbrace{\begin{bmatrix} 0 \\ \cos t \end{bmatrix}}^{g(t)}$ the same A as §7.9 #3

$$\vec{u}'(t) = \Psi^{-1} g = \frac{1}{5} \begin{bmatrix} \cos t - 2 \sin t & 5 \sin t \\ 2 \cos t + \sin t & -5 \cos t \end{bmatrix} \begin{bmatrix} 0 \\ \cos t \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \sin t \cos t \\ -5 \cos^2 t \end{bmatrix} = \begin{bmatrix} \sin t \cos t \\ -\cos^2 t \end{bmatrix}$$

$$u_1 = \int \sin t \cos t \, dt = \frac{1}{2} \sin^2 t + C_1$$

$$u_2 = \int -\cos^2 t \, dt = -\int \frac{1 + \cos 2t}{2} \, dt = -\frac{1}{2} t - \frac{1}{4} \sin 2t + C_2$$

$$\therefore \vec{u}(t) = \underbrace{\begin{bmatrix} \frac{1}{2} \sin^2 t \\ -\frac{1}{2} t - \frac{1}{4} \sin 2t \end{bmatrix}}_{\vec{v}(t)} + C \quad \Psi = \begin{bmatrix} 5 \cos t & 5 \sin t \\ 2 \cos t + \sin t & 2 \sin t - \cos t \end{bmatrix}$$

$$\text{Particular sol: } \vec{x}_p(t) = \Psi(t) \vec{v}(t) = \begin{bmatrix} \frac{5}{2} \cos t \sin t - \cos^2 t + \frac{5}{2} t + 1 \\ \cos t \sin t - \frac{1}{2} \cos^2 t + t + \frac{1}{2} \end{bmatrix}$$

$$\text{general sol: } \vec{x}(t) = \Psi(t) \vec{u}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$C_1 \begin{bmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{bmatrix} + C_2 \begin{bmatrix} 5 \sin t \\ 2 \sin t - \cos t \end{bmatrix}$$