Math 022B Final Dec., 2008

Name:__________________ ID ___________ Signature________________

No Calculators allowed. Use the back of all the pages if needed.

(1) (24pts) a)[6pts]Determine the interval in which a solution to the following Bernoulli equation is certain to exist:

$$y' + \frac{1}{t}y = \frac{1}{t^2}y^3, \quad y(1) = \sqrt{\frac{3}{2}}.$$

b)[6pts]Show that the substitution $v = y^{-2}$ reduces the above equation into a linear equation

$$v' - \frac{2}{t}v = \frac{-2}{t^2}.$$

c)[12pts]Solve the equation in (b), then solve the initial value problem in (a).

(2) (18pts) For $-1 < a < 1$, consider the ODE

$$y' = (1 - y)(y^2 - a).$$

(a)[5pts]Find all equilibrium solutions in three different cases:$(i)-1 < a < 0$, $(ii)a = 0$, $(iii)0 < a < 1$.

(b)[10pts]Sketch the phase line and determine the stability of the equilibria you found in (a) in each case.

(c)[3pts]What is the asymptotical behavior of solution $y(t)$ that satisfies the initial condition $y(0) = 0$ in each case?

(3) (20pts) a)[5pts]Show that the following differential equation is not exact:

$$dx + (x/y - \sin y)dy = 0.$$

b)[15pts]It is shown that if $(N_x - M_y)/M = Q(y)$, where $Q(y)$ is a function of y only, then the differential equation $M(x, y)dx + N(x, y)dy = 0$ can be converted into an exact equation by multiplying an integrating factor $\mu(y) = e^{\int Q(y)dy}$. Use this fact to make the differential equation in a) exact and then solve it.

(4) (28pts) a)[10pts]Find the appropriate form of a particular solution to the following differential equation and do not solve:

$$y'' - 2y' + 5y = te^{-t} + e^t \cos 2t + 3t.$$

b)[18pts]Verify that $\{y_1 = 1 + t,\ y_2 = e^t\}$ form a set of fundamental solutions to the homogeneous equation corresponding to

$$ty'' - (1 + t)y' + y = t^2 e^t,\ t > 0.$$

Then use the method of variation of parameters to find a particular solution to the non-homogeneous equation.

(5) (15pts) (a)[9pts]Find the general solution of $2y'' + y' = 0$.

(b)[6pts]Show that the solution approaches some constant as $t \to \infty$ and find the value of that constant when the solution satisfies $y(0) = 1,\ y'(0) = 1$.

(6) (24pts) Use the Laplace transform to solve the initial value problem:

$$y^{(4)} - y = 0, \quad y(0) = 1,\, y'(0) = 1, y''(0) = 1, y'''(0) = 0.$$

(Hint: write your $\mathcal{L}\{y\}(s)$ into partial fractions and use $\mathcal{L}\{\cosh(\alpha t)\} = \frac{s}{s^2-\alpha^2}$, $s > |\alpha|$, $\mathcal{L}\{\sinh(\alpha t)\} = \frac{\alpha}{s^2-\alpha^2}$, $s > |\alpha|$, $\mathcal{L}\{\sin(\alpha t)\} = \frac{\alpha}{s^2+\alpha^2}$, $s > 0$ to find the inverse transform)

(7) (19pts) Solve the following initial value problem and describe the behavior of your solution as $t \to \infty$:

$$\vec{x}\,' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(8) (24pts) a)[18pts]Find the general solution of the following system of equations:

$$\vec{x}\,' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

b)[6pts]For $\vec{x}\,' = \mathbf{A}\vec{x}$, if you have found an eigenvalue r of $\mathbf{A}$ with algebraic multiplicity 2, and there's only one eigenvector $\vec{\xi}$ associated with it, then what is the equation that the generalized eigenvecotor $\vec{\eta}$ should satisfy?

(9) (28pts) Consider the following non-homogeneous system of first order differential equation

$$\vec{x}\,' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

a)[18pts]Find the general solution to the homogeneous equation and sketch the phase portrait.

b)[4pts]If you are supposed to find a particular solution of the non-homogeneous system by method of undetermined coefficient, what is the correct form of the particular solution that you should assume?

c)[6pts]To find a particular solution of the non-homogeneous system by method of variation of parameters, we start with assuming $\vec{x}(t) = \Psi(t)\vec{u}(t)$, where $\Psi(t)$ is the fundamental matrix of the homogeneous system and $\vec{u}(t)$ is unknown. Find $\Psi(t)$ and derive the equation that $\vec{u}(t)$ should satisfy. You don't have to solve them.