Math 022B Midterm 2 Feb. 26, 2010

Name: Solution ID

No Calculators allowed. Use the back of all the pages if needed.

(1) (16 points) (a) Find the solution $y(t)$ of the initial value problem

$$4y'' - y = 0, \quad y(0) = 2, \ y'(0) = \beta.$$

(b)Find β so that the solution approaches zero as $t \to \infty$.

(a) $4r^2 - 1 = 0 \Rightarrow r = \pm\frac{1}{2}$ (4) $\Rightarrow y = c_1 e^{-t/2} + c_2 e^{+t/2}$ (3)

initial conditions imply: $c_1 + c_2 = 2 \quad -\frac{1}{2}c_1 + \frac{1}{2}c_2 = \beta$ (2)

$\Rightarrow c_1 = 1-\beta, \ c_2 = 1+\beta$ (2) $\Rightarrow y = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$ (1)

(b) $y \to 0$ as $t \to \infty$ if $1+\beta = 0$, i.e. $\beta = -1$ (4)

(2) (18 points) Suppose that the position of a certain spring-mass system satisfies the initial value problem:

$$\frac{3}{2}u'' + ku = 0, \ u(0) = 2, \ u'(0) = v. \qquad k>0$$

Determine the values of k and v if the period and amplitude of the resulting motion are π and 3, respectively. *(Hint: for a general equation: $mu'' + ku = 0$, denote $\omega_0 = \sqrt{\frac{k}{m}}$, then period $T = \frac{2\pi}{\omega_0}$, and amplitude $R = \sqrt{A^2+B^2}$, if the solution is $u = A\cos\omega_0 t + B\sin\omega_0 t$.)* $m, k > 0$

$m = \frac{3}{2}$, $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \Rightarrow k = m\cdot\left(\frac{2\pi}{T}\right)^2 = \frac{3}{2}\cdot\left(\frac{2\pi}{\pi}\right)^2 = 6$ (6)

The solution will indeed be of the form $u(t) = A\cos\omega_0 t + B\sin\omega_0 t$ (+)

initial conditions imply: $\begin{cases} u(0) = A = 2 \\ u'(0) = B\omega_0 = v \end{cases}$ so $\begin{cases} A = 2 \\ B = \frac{v}{\omega_0} = \frac{v}{2} \end{cases}$ (+)

$R = \sqrt{A^2+B^2} \Rightarrow 3 = \sqrt{2^2 + \left(\frac{v}{2}\right)^2} \Rightarrow v = \pm 2\sqrt{5}$ (4)

(3) (7+4+13 points) (a) Let $y_1(t) \neq 0$ be a known solution of $L[y] = y'' + p(t)y' + q(t)y = 0$, show that a 2nd solution y_2 satisfies $\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \frac{W(y_1, y_2)}{y_1^2}$, where $W(y_1, y_2) = y_1' y_2 - y_1 y_2'$.

(b) State the Abel's formula for finding $W(y_1, y_2)$ of $L[y] = 0$ in (a).

(c) Use (a) and (b) to find a second solution for

$$t^2 y'' + 3ty' + y = 0, \ t > 0; \quad y_1(t) = t^{-1}$$

(a) $\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \dfrac{\frac{dy_2}{dt} y_1 - y_2 \frac{dy_1}{dt}}{y_1^2} = \dfrac{W(y_1, y_2)}{y_1^2}$ (7)

(b) $W(y_1, y_2) = c\exp\left(\int p(t)\, dt\right)$ (4)

(c) $y'' + \frac{3}{t} y' + \frac{1}{t^2} y = 0$

$\Rightarrow p = \frac{3}{t} \Rightarrow W(y_1, y_2) = c \cdot \exp\left(-\int \frac{3}{t} dt\right) = c \cdot \frac{1}{t^3}$ (5)

$\Rightarrow \frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \dfrac{c \cdot \frac{1}{t^3}}{(t^{-1})^2} = c \cdot \frac{1}{t}$ (4)

$\Rightarrow \frac{y_2}{y_1} = c \ln t \Rightarrow y_2 = c\, y_1 \ln t = c\, \frac{\ln t}{t}$ (4)

You may choose $c = 1$ so that $y_2 = \frac{\ln t}{t}$

(4) (16 points) Use the Laplace transform to solve the following initial value problem:

$$y^{(4)} - y = 0, \quad y(0) = 1, \ y'(0) = 0, \ y''(0) = 1, \ y'''(0) = 0.$$

(*Hint: you may need* $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$ *to do the inverse Laplace transform.*)

$\mathcal{L}\{y^{(4)} - y\} = 0 \Rightarrow s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - \mathcal{L}\{y\} = 0$ (11)

$\Rightarrow \mathcal{L}\{y\} = \dfrac{s^3 + s}{s^4 - 1} = \dfrac{s(s^2+1)}{(s^2+1)(s^2-1)} = \dfrac{s}{s^2 - 1}$ (4)

$\Rightarrow y = \cosh t \quad (a = 1)$ (1)

(5) (10+ 16 points) (a) Determine a suitable form for a particular solution $Y(t)$ (Do NOT solve!) for the following equation if the method of undetermined coefficients is to be used:

$$y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-t}\cos t$$

(b) Use the method of variation of parameters to find a particular solution of $y'' + 2y' + y = 3e^{-t}$.

(a) $r^2+2r+5=0 \Rightarrow r=-1\pm 2i \Rightarrow t^{-t}\cos 2t$ is a sol of the homogeneous eqn. So the suitable form of $Y(t)$ would be

$$t\left[(A_1t+B_1)e^{-t}\cos 2t + (A_2t+B_2)e^{-t}\sin 2t\right]$$ 4 4

$$-\left[(A_3t+B_3)e^{-t}\cos t + (A_4t+B_4)e^{-t}\sin t\right]$$ 2

(b) $r^2+2r+1=0 \Rightarrow r_{1,2}=-1 \Rightarrow y(t)=C_1e^{-t}+C_2te^{-t}$

Let $Y(t)=u_1(t)e^{-t}+u_2(t)te^{-t}$ be a particular sol 4

$$\Rightarrow \begin{cases} u_1'\cdot e^{-t} + u_2'\cdot te^{-t} = 0 \\ u_1'\cdot(-e^{-t}) + u_2'\cdot e^{-t}(1-t) = 3e^{-t} \end{cases}$$ 4

$$\Rightarrow \begin{cases} u_1' = -3t \\ u_2' = 3 \end{cases} \Rightarrow \begin{cases} u_1(t) = -\frac{3}{2}t^2 + C_1 \\ u_2(t) = 3t + C_2 \end{cases}$$ 6

$$\Rightarrow Y(t) = -\frac{3}{2}t^2e^{-t} + 3t\cdot te^{-t} + C_1e^{-t} + C_2te^{-t}$$

$$= \frac{3}{2}t^2e^{-t} + C_1e^{-t} + C_2te^{-t}$$ 2

So we may choose $C_1=C_2=0$ and $Y(t)=\frac{3}{2}t^2e^{-t}$ is a particular solution

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OR if you directly use the following formula:

$$Y = -y_1\int \frac{y_2 g}{W}\,dt + y_2\int\frac{y_1 g}{W}\,dt$$ 10

$$\Rightarrow Y = \frac{3}{2}t^2e^{-t} + C_1e^{-t} + C_2te^{-t}$$ 6