Answer Key

Math 022B | Final | Dec. 9, 2008

Name:________________ ID __________ Signature______________

No Calculators allowed. Use the back of all the pages if needed.

(1) (24pts) a)[6pts]Determine the interval in which a solution to the following Bernoulli equation is certain to exist:

$$y' + \frac{1}{t}y = \frac{1}{t^2}y^3, \quad y(1) = \sqrt{\frac{3}{2}}.$$

b)[6pts]Show that the substitution $v = y^{-2}$ reduces the above equation into a linear equation

$$v' - \frac{2}{t}v = \frac{-2}{t^2}.$$

c)[12pts]Solve the equation in (b), then solve the initial value problem in (a).

Sol:

(a) rewrite as $y' = f(t,y) = -\frac{1}{t}y + \frac{1}{t^2}y^3$

By existence and uniqueness theorem, you need to find the interval, containing $t_0=1$, in which both f and $\frac{\partial f}{\partial y}$ are continous. Clearly $t=0$ is the only discontinuous point, so $\boxed{(0, +\infty)}$ is the interval

(b) $v = y^{-2} \Rightarrow y = \pm v^{-\frac{1}{2}} \Rightarrow y' = \pm(-\frac{1}{2})v^{-\frac{3}{2}} \cdot v'$ ← Note: Actually only "+" is possible by the initial data

then $\pm\left[(-\frac{1}{2})v^{-\frac{3}{2}}v' + \frac{1}{t}v^{-\frac{1}{2}}\right] = \frac{1}{t^2}(\pm v^{-\frac{1}{2}})^3$

the "±" sign will be cancelled, then multiply both sides by $-2 \cdot v^{\frac{3}{2}}$

we get $v' - \frac{2}{t}v = -\frac{2}{t^2}$

OR: you can follow the solution from HW#2, simply divide $-\frac{1}{2}y^3$ on both sides of $y' + \frac{1}{t}y = \frac{1}{t^2}y^3$, recognising $-2\frac{y'}{y^3} = v'$, then we are done.

(c) Integrating factor $\mu(t) = e^{-\int \frac{2}{t}dt} = ce^{-2\ln t} = c \cdot t^{-2}$

choosing $c=1$, then $[v \cdot \frac{1}{t^2}]' = -2t^{-4}$

$\Rightarrow \frac{v}{t^2} = \frac{2}{3}t^{-3} + C \Rightarrow \boxed{v = \frac{2}{3t} + ct^2}$

Initial value $y(1) = \sqrt{\frac{3}{2}} \Rightarrow v(1) = (y(1))^{-2} = \frac{2}{3} \Rightarrow C=0 \Rightarrow v = \frac{2}{3t}$

so $\boxed{y = +\sqrt{\frac{1}{v}} = \sqrt{\frac{3t}{2}}}$. Note the "−" sign is not possible, since $y(1) = \sqrt{\frac{3}{2}} > 0$

(2) (18pts) For $-1 < a < 1$, consider the ODE

$$y' = (1-y)(y^2-a),$$

(a)[5pts]Find all equilibrium solutions in three different cases:$(i) -1 < a < 0$, $(ii) a = 0$, $(iii) 0 < a < 1$.

(b)[10pts]Sketch the phase line and determine the stability of the equilibria you found in (a) in each case.

(c)[3pts]What is the asymptotical behavior of solution $y(t)$ that satisfies the initial condition $y(0) = 0$ in each case?

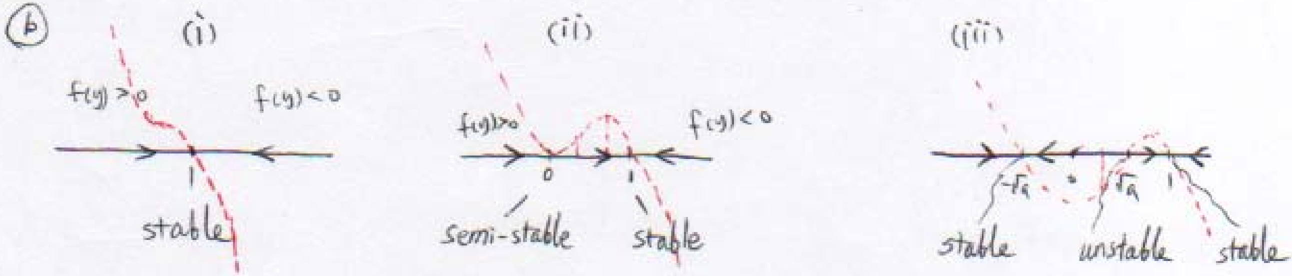
Sol: $y' = f(y) = (1-y)(y^2-a)$

(a) equilibrium solutions are the y^*'s such that $f(y^*) = 0$

$(1-y^*)(y^{*2}-a) = 0 \Rightarrow y^* = 1$ or $y^{*2} = a$

So in case (i) $y^* = 1$ (ii) $y^* = 1$ or $y^* = 0$ (iii) $y^* = 1$ or $y^* = \pm\sqrt{a}$

(b)



Note: you don't need to draw the f(y) curve, but you can still tell the signs of f(y), i.e, y', on each intervals

(c) By the analysis we have in (b), $y \to y^* = 1$ in case (i)

$y \equiv 0$ in case (ii) / since $y^* = 0$ is an equil sol.

$y \to -\sqrt{a}$ in case (iii)

(3) (20pts) a)[5pts]Show that the following differential equation is not exact:

$$dx + (x/y - \sin y)dy = 0$$

b)[15pts]It is shown that if $(N_x - M_y)/M = Q(y)$, then the differential equation $M + Ny' = 0$ can be converted into an exact equation by multiplying an integrating factor $\mu(y) = e^{\int Q(y)dy}$. Use this fact to make the differential equation in a) exact and then solve it.

Sol: (a) $M = 1 \quad N = x/y - \sin y$

Exactness $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ / OR $M_y = N_x$

So $\frac{\partial M}{\partial y} = 0 \neq \frac{\partial N}{\partial x} = \frac{1}{y} \Rightarrow$ Not exact

(b) Here $(N_x - M_y)/M = (\frac{1}{y} - 0)/1 = \frac{1}{y}$

$\Rightarrow$ we have $\mu(y) = e^{\int \frac{1}{y} dy} = c \cdot y$

choose $c = 1$

then $y\,dx + y(x/y - \sin y)\,dy = 0$ is exact.

i.e. $y\,dx + (x - y\sin y)\,dy = 0$

So $\tilde{M} = (\mu M) = y \quad , \quad \tilde{N} = (\mu N) = x - y\sin y$

$\tilde{M} = \phi_x \Rightarrow \phi(x,y) = \int \tilde{M}\,dx = \int y\,dx = xy + h(y)$

We also need $\phi_y = \tilde{N} \Rightarrow x + h'(y) = x - y\sin y$

$\Rightarrow h'(y) = -y\sin y \Rightarrow h(y) = -\int y\sin y\,dy$

$= y\cos y - \int \cos y\,dy$

$= y\cos y - \sin y + c$

Hence $\phi(x,y) = xy + y\cos y - \sin y + c$

i.e. the general (implicit) solution

is $\boxed{xy + y\cos y - \sin y = C}$

(4) (28pts) a)[10pts]Find the appropriate form of a particular solution to the following differential equation and do not solve.

$$y'' - 2y' + 5y = te^{-t} + e^t \cos 2t + 3t$$

b)[18pts]Verify that $\{y_1 = 1 + t,\ y_2 = e^t\}$ form a set of fundamental solutions to the homogeneous equation corresponding to

$$ty'' - (1 + t)y' + y = t^2 e^t,\ t > 0.$$

Then use the method of variation of parameters to find a particular solution to the non-homogeneous equation.

Sol: (a) check the roots of the characteristic equation first:

$r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i \Rightarrow y_c = e^t(c_1 \cos 2t + c_2 \sin 2t)$ for homogeneous eqn

$g_1 = te^{-t} \rightarrow Y_1 = (A_0 t + A_1)e^{-t}$

$g_2 = e^t \cos 2t \rightarrow Y_2 = t\cdot(A_2 \cos 2t + A_3 \sin 2t)e^t$

$g_3 = 3t \rightarrow Y_3 = A_4 t + A_5$

Note: you only need to multiply t in Y_2 since $e^t \cos 2t$ is a sol to the homogeneous equation

So the appropriate form of a particular sol

is $Y(t) = (A_0 t + A_1)e^{-t} + t\cdot(A_2 \cos 2t + A_3 \sin 2t)e^t + A_4 t + A_5$

(b) $y_1 = 1+t \Rightarrow y_1' = 1,\ y_1'' = 0 \Rightarrow t\cdot 0 - (1+t)\cdot 1 + (1+t) = 0$

$y_2 = e^t \Rightarrow y_2' = y_2'' = e^t \Rightarrow e^t(t - (1+t) + 1) = 0$

$\Rightarrow y_1, y_2$ are solutions of the homogeneous eqn.

Further, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_1' \cdot y_2 = (1+t)e^t - e^t = t\cdot e^t \neq 0$, since $t > 0$.

Hence $\{y_1, y_2\}$ form a set of fundemental solutions to $ty'' - (1+t)y' + y = 0$

Then let $y(t) = (1+t)u_1(t) + e^t \cdot u_2(t)$

rewrite the non-homogeneous eqn as $y'' - (1 + \frac{1}{t})y' + \frac{1}{t}y = te^t$ ($g(t)$)

then

$$\begin{cases} u_1' \cdot (1+t) + u_2' \cdot e^t = 0 & ① \\ u_1' \cdot 1 + u_2' \cdot e^t = t\cdot e^t & ② \end{cases}$$

$① - ② \Rightarrow u_1' = -e^t$, so $u_2' = (1+t)$

$\Rightarrow u_1 = -e^t + c_1,\quad u_2 = t + \frac{t^2}{2} + c_2$

$\Rightarrow y(t) = (1+t)(-e^t + c_1) + (t + \frac{t^2}{2} + c_2)e^t = (\frac{t^2}{2} - 1)e^t + c_1(1+t) + c_2 e^t$

$\Rightarrow \boxed{Y(t) = (\frac{t^2}{2} - 1)e^t}$ is a particular solution.

(5) (15pts) (a)[9pts]Find the general solution of $2y'' + y' = 0$.

(b)[6pts]Show that the solution approaches some constant as $t \to \infty$ and find the value of that constant when the solution satisfies $y(0) = 1$, $y'(0) = 1$.

sol: Characteristic eqn: $2r^2 + r = 0 \Rightarrow r = 0$ or $r = -\frac{1}{2}$

(a) $\Rightarrow$ general sol. $y(t) = C_1 + C_2 e^{-\frac{1}{2}t}$

(b) $e^{-\frac{t}{2}} \to 0$ as $t \to \infty$, so $\boxed{y(t) \to C_1}$

$$\begin{cases} y(0) = C_1 + C_2 = 1 \\ y'(0) = -\frac{1}{2} C_2 = 1 \end{cases} \Rightarrow \boxed{C_1 = 3}$$

(6) (24pts) Use the Laplace transform to solve the initial value problem:

$$y^{(4)} - y = 0, \quad y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 0.$$

(Hint: write your $\mathcal{L}\{y\}(s)$ into partial fractions and use $\mathcal{L}\{\cosh(\alpha t)\} = \frac{s}{s^2-\alpha^2}$, $s > |\alpha|$, $\mathcal{L}\{\sinh(\alpha t)\} = \frac{\alpha}{s^2-\alpha^2}$, $s > |\alpha|$, $\mathcal{L}\{\sin(\alpha t)\} = \frac{\alpha}{s^2+\alpha^2}$, $s > 0$ to find the inverse transform)

Sol: $\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0$

$$\Rightarrow \quad s^4\mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s\, y''(0) - y'''(0) - \mathcal{L}\{y\} = 0$$

$$\Rightarrow \quad (s^4-1)\mathcal{L}\{y\} = s^3 + s^2 + s$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{s^3+s^2+s}{s^4-1} = \frac{s(s^2+1)+s^2}{(s^2+1)(s^2-1)} = \frac{s}{s^2-1} + \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{1}{s^2-1}\right)$$

So $\boxed{y = \cosh t + \frac{1}{2}(\sinh t + \sin t)}$

(7) (19pts) Solve the following initial value problem and describe the behavior of your solution as $t \to \infty$:

$$\mathbf{X}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Sol: $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$ $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$

for $\lambda = -1+i$, $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \xi_1 = (2+i)\xi_2$

$\Rightarrow \vec{\xi}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \Rightarrow \vec{\xi}_1 e^{\lambda t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$ is a solution.

From: $\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t} \cdot (\cos t + i \sin t)$

$= e^{-t} \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ 1 \cdot \cos t + i \sin t \end{pmatrix}$

$= e^{-t} \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \right]$

We get the (real) general solution:

$$\vec{X}(t) = e^{-t} \left[C_1 \cdot \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \right]$$

$\vec{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow C_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

So $\vec{X}(t) = e^{-t} \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \right] = \boxed{e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}}$

We can see that $\boxed{\lim_{t \to \infty} \vec{X}(t) = \vec{0}.}$ Both $x_1(t)$ and $x_2(t)$ oscillates near $y=0$ and approach $y=0$

(8) (24pts) a)[18pts]Find the general solution of the following system of equations:

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{X}$$

b)[6pts]For a 3x3 matrix **A**, if you have found an eigenvalue r with algebraic multiplicity 2, and there's only one eigenvector $\vec{\xi}$ associated with it, then what is the equation that the generalized eigenvecotor $\vec{\eta}$ should satisfy?

Sol: (a) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad \det(A-\lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (-\lambda)^3 + 1 + 1 - 3(-\lambda) = 0 \quad \Rightarrow \quad \lambda^3 - 3\lambda - 2 = 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_{2,3} = -1$$

For $\lambda_1 = 2$ $\quad \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0 \Rightarrow \xi_1 = \xi_2 = \xi_3 \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda_{2,3} = -1$ $\quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0 \Rightarrow \xi_1 + \xi_2 + \xi_3 = 0 \Rightarrow \begin{cases} \vec{\xi}_2 = \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} \\ \vec{\xi}_3 = \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix} \end{cases}$

$\Rightarrow$ general solution: $\quad \vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t}$

(b) let $\vec{x} = \vec{\xi} t e^{rt} + \vec{\eta} e^{rt}$, then $\vec{\eta}$ is the generalized eigenvector

Substitute into $\vec{x}' = A\vec{x} \Rightarrow \begin{cases} (A - rI)\vec{\xi} = 0 \\ (A - rI)\vec{\eta} = \vec{\xi} \end{cases}$

So $\boxed{(A - rI)\vec{\eta} = \vec{\xi}}$ or $\boxed{(A - rI)^2 \vec{\eta} = 0}$

(9) (28pts) Consider the following non-homogeneous system of first order differential equation

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

a)[18pts]Find the general solution to the homogeneous equation and sketch the phase portrait.

b)[4pts]If you are supposed to find a particular solution of the non-homogeneous system by method of undetermined coefficient, what is the correct form of the particular solution that you should assume?

c)[6pts]To find a particular solution of the non-homogeneous system by method of variation of parameters, we start with assuming $\mathbf{X}(t) = \Psi(t)\mathbf{u}(t)$, where $\Psi(t)$ is the fundamental matrix of the homogeneous system and $\mathbf{u}(t)$ is unknown. Find $\Psi(t)$ and derive the equations that $\mathbf{u}(t)$ should satisfy. You don't have to solve them.

Sol. ⓐ $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$

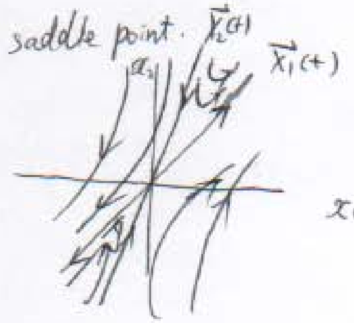
$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1$

For $\lambda_1 = 1$ $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \xi_1 = \xi_2 \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -1$ $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow 3\xi_1 = \xi_2 \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\Rightarrow \boxed{\vec{X}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}}$. Origin is a saddle point.

phase portrait:



ⓑ $\vec{g}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ The correct form of a particular sol is

$\boxed{\vec{X}(t) = \vec{a}\, t e^t + \vec{b} e^t + \vec{c}\, t + \vec{d}}$

Note: We need this form because $\vec{k} e^t$ is a solution of the homogeneous system.

ⓒ $\Psi(t) = \begin{pmatrix} \vec{X}_1 & \vec{X}_2 \end{pmatrix} = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$

$\vec{X}(t) = \Psi \vec{u} \Rightarrow \vec{X}' = \Psi' \cdot \vec{u} + \Psi \cdot \vec{u}' = A \Psi \vec{u} + \vec{g}$

$\Psi' = A\Psi$ by property of the fundamental matrix so $\Psi \cdot \vec{u}' = \vec{g}$

In this perticular case: $\boxed{\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \\ t \end{pmatrix}}$.