Math 022B Midterm 2 Nov. 21, 2008

Name: Solution ID ____________

No Calculators allowed. Use the back of all the pages if needed.

(1) (16 points) (a) Find the solution $y(t)$ of the initial value problem

$$4y'' - y = 0, \quad y(0) = \alpha, \; y'(0) = 2.$$

(b) Find α so that the solution approaches zero as $t \to \infty$.

(a) $4r^2 - 1 = 0 \Rightarrow r_1 = \frac{1}{2} \quad r_2 = -\frac{1}{2} \Rightarrow y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-\frac{1}{2}t}$

initial conditions imply: $\begin{cases} C_1 + C_2 = \alpha \\ \frac{1}{2}C_1 - \frac{1}{2}C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = 2 + \frac{\alpha}{2} \\ C_2 = \frac{\alpha}{2} - 2 \end{cases}$

$\Rightarrow y(t) = (2 + \frac{\alpha}{2})e^{\frac{1}{2}t} + (\frac{\alpha}{2} - 2)e^{-\frac{1}{2}t}$

(b) we need $2 + \frac{\alpha}{2} = 0$, so $\alpha = -4$

(2) (22 points) (a) Verify that $y_1(t) = t^2$ is a solution of the ODE

$$t^2 y'' - 2y = 0 \, (t > 0).$$

(b) Use the method of reduction of order to find a second solution $y_2(t)$ and verify that $\{y_1, y_2\}$ form a fundamental set of solutions of the ODE.

(a) $y_1 = t^2 \Rightarrow y_1' = 2t \quad y_1'' = 2$

$t^2 \cdot 2 - 2 \cdot t^2 = 0 \Rightarrow y_1 = t^2$ is a sol of the ODE.

(b) let $y_2 = v(t) t^2$ be another sol of the ODE.

then $y_2' = v' \cdot t^2 + v \cdot 2t$

$y_2'' = v'' \cdot t^2 + 2 \cdot (2v't) + 2v$

$\Rightarrow t^2 v'' + 4t v' = 0$, Treat $v' = w$ we can solve

for $w_1 = C_1 \cdot t^{-4}$ then $v = \int w \, dt = C_2 t^{-3} + C_3$

$\Rightarrow \boxed{y_2 = t^{-3} \cdot t^2 = \frac{1}{t}}$

$W(t^2, t^{-1}) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0$

Hence $\{y_1, y_2\}$ form a fundamental set of solutions.

(3) (26 points) (a) Find a particular solution of the ODE by the method of undetermined coefficients:

$$y'' + 2y' + y = 3te^{-t}$$

(b)Find the general solution.

(c)Re-do part (a) by the method of variation of parameters.

(a) Solve the homogeneous eq first: $r^2+2r+1=0 \Rightarrow r_{1,2}=-1 \Rightarrow e^{-t}, te^{-t}$ are both solutions of the homogeneous eqn.

Then let $Y(t) = t^2(A_1 t + A_2)e^{-t}$ be a particular solution

We have $Y' = e^{-t}(3A_1t^2 + 2A_2 t - A_1 t^3 - A_2 t^2) = e^{-t}(-A_1t^3 + (3A_1 - A_2)t^2 + 2A_2 t)$

$Y'' = e^{-t}(-3A_1t^2 + 2(3A_1 - A_2)t + 2A_2 + A_1t^3 - (3A_1 - A_2)t^2 - 2A_2 t)$

$= e^{-t}(A_1 t^3 + (-6A_1 + A_2)t^2 + (6A_1 - 4A_2)t + 2A_2)$

Substitute Y, Y' & Y'' into the non-homogeneous eqn: $\boxed{6A_1 t + 2A_2 = 3t}$

$\Rightarrow A_1 = \frac{1}{2}, \; A_2 = 0$ i.e. $\boxed{Y(t) = \frac{1}{2}t^3 e^{-t}}$ is a particular sol.

(b) $Y_c(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}t^3 e^{-t}$

(c) let $Y(t) = u_1 e^{-t} + u_2 t e^{-t}$

then
$$\boxed{\begin{cases} u_1' e^{-t} + u_2' t e^{-t} = 0 \\ u_1' \cdot (-e^{-t}) + u_2'(1-t)e^{-t} = 3te^{-t} \end{cases}}$$

$$\Rightarrow \begin{cases} u_1' = -3t^2 \\ u_2' = 3t \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -t^3 + c_1 \\ u_2 = \frac{3}{2}t^2 + c_2 \end{cases}$$

$\Rightarrow Y(t) = (-t^3 + c_1)e^{-t} + (\frac{3}{2}t^2 + c_2)t \cdot e^{-t}$

$= \boxed{\frac{1}{2}t^3 e^{-t}} + c_1 e^{-t} + c_2 t e^{-t}$

(4) (16 points) Use the Laplace transform to solve the initial value problem:

$$y'' + 2y' + y = 4e^{-t}, \quad y(0) = 2,\ y'(0) = -1.$$

(Hint: you can break your function into several parts and use $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ to do the inverse Laplace transform)

sol: $\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{4e^{-t}\}$

$\Rightarrow \quad s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{y\} = 4\cdot\frac{1}{s+1}$

$\Rightarrow \quad (s^2+2s+1)\mathcal{L}\{y\} - (s+2)y(0) - y'(0) = \frac{4}{s+1}$

$\Rightarrow \quad \mathcal{L}\{y\} = \frac{(s+2)\cdot 2 - 1}{s^2+2s+1} + \frac{4}{(s+1)(s^2+2s+1)} = \frac{2s+3}{(s+1)^2} + \frac{4}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$

$\Rightarrow \quad y = 2\cdot e^{-t} + t^1\cdot e^{-t} + 2\cdot t^2\cdot e^{-t} = \boxed{e^{-t}(2t^2+t+2)}$

(5) (20 points) (a) Find the general solution of the following system of equations and sketch the phase portrait of the system.

$$\mathbf{X}' = \begin{pmatrix} 1/2 & 3/4 \\ 3 & 1/2 \end{pmatrix}\mathbf{X}$$

(b) Find the solution that satisfies the initial condition:

$$\mathbf{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$