Mat127a Discussion Section 1

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Exercise 1.1 & 1.2 [We prove **Exercise 1.3** instead.] In view of Example 1, we may write our n-th proposition as

$$P_n : 1^3 + 2^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

The basis for induction P_1 is clearly true, since $1^3 = 1^2$. For the induction step, suppose that P_n is true. We then have

$$\begin{aligned} & 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \\ &= \left[\frac{1}{2}n(n+1)\right]^2 + (n+1)^3 \\ &= \left[\frac{1}{2}(n+1)\right]^2 [n^2 + 4(n+1)] \\ &= \left[\frac{1}{2}(n+1)\right]^2 (n+2)^2 \\ &= \left[\frac{1}{2}(n+1)(n+2)\right]^2, \end{aligned}$$

so P_{n+1} is also true and the induction step holds. Therefore, P_n is true for all natural numbers n.

Question I'm pretty sure that all of you know how to show the identity $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ directly without using induction. Can you do the same with the identities in Exercise 1.1 & 1.3? Formulas like $(n+1)^2 - n^2 = 2n+1$ and $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ would be helpful. Another example where direct proof exists and is more preferred is Exercise 1.12(c). For this you may want to take Math145.

Exercise 1.7 Our n-th proposition is

$$P_n : 7^n - 6n - 1 \text{ is divisible by } 36.$$

The basis for induction P_1 is clearly true, since $7^1 - 6\cdot 1 - 1 = 0$ is divisible by 36. For the induction step, suppose that P_n is true. We write

$$7^{n+1} - 6(n+1) - 1 = \underline{\qquad}(7^n - 6n - 1) + \underline{\qquad\quad}$$

and observe $7^{n+1} - 6(n+1) - 1$ is also divisible by 36. So P_{n+1} is true and the induction step holds. Therefore, P_n is true for all natural numbers n. [You might want to see page 311.]

Exercise 1.11 (a) See page 312.

(b) Because $n(n+1)$ is even for all n, we see that $n^2 + 5n + 1 = \underline{\qquad\qquad} + 4\underline{\quad} + 1$ is odd for all n. That is, P_n is false for all n. The moral here is that the basis for induction **(I$_1$)** is crucial for mathematical induction.

Exercise 1.12 (a) We all know the following identities:

$$\begin{aligned} (a+b)^1 &= a^1 + b^1 \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3. \end{aligned}$$

(b) First observe that all three terms involved are defined for $k = 1, 2, \ldots, n$. We then verify

$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= \frac{(n+1)!}{k!(n+1-k)!} \\ &= \binom{n+1}{k}. \end{aligned}$$

(This identity is called *Pascal's formula*.)

(c) Using summation notation, our n-th proposition is

$$P_n : (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

In (a) we observed that P_1 is true. For the induction step, suppose that P_n is true. By (b), we then have

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} \\
&= a^{n+1} + \sum_{\underline{\quad}}^{\underline{\;}}\binom{n}{k}a^{n+1-k}b^k \\
&\qquad + \sum_{\underline{\quad}}^{\underline{\quad}}\binom{n}{k}a^{n-k}b^{k+1} + b^{n+1} \\
&= a^{n+1} + \sum_{\underline{\quad}}^{\underline{\;}}\binom{n}{k}a^{n+1-k}b^k \\
&\qquad + \sum_{\underline{\quad}}^{\underline{\;}}\binom{n}{k-1}a^{n+1-k}b^k + b^{n+1} \\
&= a^{n+1} + \sum_{\underline{\quad}}^{\underline{\;}}\left[\binom{n}{k}+\binom{n}{k-1}\right]a^{n+1-k}b^k + b^{n+1} \\
&= a^{n+1} + \sum_{\underline{\quad}}^{\underline{\;}}\binom{n+1}{k}a^{n+1-k}b^k + b^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k,
\end{aligned}
$$

so P_{n+1} is also true and the induction step holds. Therefore, P_n is true for all natural numbers n.

Exercise 2.1 (i) If $\sqrt{3}$ were a rational number, then it would be a solution of $x^2-3=0$. By Theorem 2.2, however, the only possible rational solutions of this equation are $\pm1, \pm3$ and it is obviuos that none of these numbers are solutions. The cases of $\sqrt{5}, \sqrt{7}, \sqrt{31}$ are similar to this, because $3, 5, 7, 31$ are prime numbers.

(ii) If $\sqrt{24}$ were a rational number, then it would be a solution of $x^2-24=0$. By Theorem 2.2, however, the only possible rational solutions of this equation are $\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ and it is easy to verify that none of these numbers are solutions.

Let's do another, say **Exercise 2.5**. The expression $a=(3+\sqrt{2})^{2/3}$ means $a^3=(3+\sqrt{2})^2=11+6\sqrt{2}$ or $a^3-11=6\sqrt{2}$ so that $(a^3-11)^2=(6\sqrt{2})^2=72$. Therefore we have $a^6-22a^3+49=0$, for which Theorem 2.2 is applicable. (Why?) The theorem says the only possible rational solutions of this equation are $\pm1, \pm7, \pm49$ and you are to verify that none of these numbers are solutions.

Exercise 3.1 (a) **A3**, **A4**, and **M4** fail. Why? (b) **M4** fails. Why?

Exercise 3.5 (a) Since $|b|$ is equal to either b or $-b$,

$$
\begin{aligned}
|b| \le a \quad &\text{if and only if} \quad \underline{\qquad\qquad\qquad\qquad} \\
&\text{if and only if} \quad b \le a \;\text{ and }\; -a \le b \\
&\text{if and only if} \quad -a \le b \le a.
\end{aligned}
$$

(b) In view of (a), it suffices to show that

$$-|a-b| \le |a|-|b| \le |a-b|$$

for all $a, b \in \mathbb{R}$. Using the triangle inequality, show the first and the second inequalities, respectively. [You might want to see page 312.]

Here's a good problem where proof by contradiction is very useful, namely **Exercise 3.8**. Suppose not, that is, $a>b$. Then $\frac{a+b}{2}>b$ but $a>\frac{a+b}{2}$; a contradiction! [Can you prove this directly?]

Exercise 4.3 First, here are the answers. (a) 1 (b) 1 (c) 7 (d) π (e) 1 (f) 0 (g) 3 (h) NO sup (i) 1 (j) 1 (k) NO sup (l) 2 (m) 2 (n) $\sqrt{2}$ (o) 0 (p) 10 (q) 16 (r) 1 (s) $1/2$ (t) 2 (u) NO sup (v) 1 (w) $\sqrt{3}/2$.

I think you are really required to present proofs in case of (i), (j), (r), at least. For example, to prove $\{1\}=\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$, you may first want to show $\{1\}\supseteq\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$. To do that, one starts like "Suppose $r\notin\{1\}$, i.e. $r>1$ or $r<1$. If $r>1$, then blah, blah..." and the Archimedean property is crucial in doing so.

Okay. Here comes an example. Let's show that

$$\bigcup_{n=1}^{\infty} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] = (-1,1).$$

The following is my solution. Your reader may ask more than this, though.

($\subseteq$) Clear. ($\supseteq$) Suppose $r \in (-1,1)$. Because $r > -1$, we have $r+1 > 0$ and by the Archimedean property there is a natural number n_1 such that $\frac{1}{n_1} < r+1$, so $-1+\frac{1}{n_1} < r$. On the other hand, since $r < 1$, we have $1-r > 0$ and one can similarly show that $1-r > \frac{1}{n_2}$ for some natural number n_2, so $r < 1-\frac{1}{n_2}$. Let $m = \max\{n_1, n_2\}$. Then we have $r \in [-1+\frac{1}{m}, 1-\frac{1}{m}]$ and hence $r \in \cup_{n=1}^{\infty}[-1+\frac{1}{n}, 1-\frac{1}{n}]$. The proof is complete.

Exercise 4.5 & 4.11 See page 313.

Exercise 5.1 & 5.5 See page 314.