Mat127a Discussion Section 4

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Exercise 11.2 (a) (a_{2n}) and the others are themselves monotonic;

(b) $\{-1, 1\}$, $\{0\}$,$\{+\infty\}$,$\{\frac{6}{7}\}$;

(c) $\liminf a_n = -1$, $\limsup a_n = 1$, $\liminf b_n = \limsup b_n = 0$, $\liminf c_n = \limsup c_n = +\infty$, $\liminf d_n = \limsup d_n = \frac{6}{7}$;

(d) (a_n) does not converge, (b_n) converges to 0, (c_n) diverges to $+\infty$, (d_n) converges to $\frac{6}{7}$;

(e) (a_n), (b_n), and (d_n) are bounded.

Exercise 11.5 (a) $S = [0, 1]$. (The proof is essentially the same as Example 3 on page 65. For example, you may want to explain why $0 \in S$.) (b) $\limsup q_n = \sup S = 1, \liminf q_n = \inf S = 0$ by Theorem ______.

Exercise 11.6 Note that a sequence t is a subsequence of a sequence s if and only if $t = s \circ \sigma$ for some increasing function $\sigma : \mathbb{N} \to \mathbb{N}$.

Now let q be a subsequence of p and let r be a subsequence of q. Then there are increasing functions σ and τ mapping $\mathbb{N}$ into $\mathbb{N}$ such that ________ and ________. Since $r =$ ________ $= p \circ ($______$)$ and $\sigma \circ \tau : \mathbb{N} \to \mathbb{N}$ is also __________ (why?), we see that r is itself a subsequence of p.

Exercise 11.10 (a) $S = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. (You need to explain this. Find subsequences which converge to 0 and $\frac{1}{n}$, respectively.) (b) By Theorem ______, we have that $\limsup s_n = \sup S = 1$ and $\liminf s_n = \inf S = 0$.

Exercise 12.2 $\lim s_n = 0 \overset{[i]}{\iff} \lim |s_n| = 0 \overset{[ii]}{\iff} \limsup |s_n| = 0$. [i] Exercise ______ [ii] Theorem ______

Exercise 12.3 Note that $(s_n + t_n) = (2, 2, 3, 1, \ldots)$ and $(s_n t_n) = (0, 1, 2, 0, \ldots)$. (a) $0 + 0$; (b) 1; (c) $0 + 2$; (d) 3; (e) $2 + 2$; (f) 0; (g) 2.

Exercise 12.5 See page 319.

Exercise 12.14 (b) Denote $s_n = \frac{n!}{n^n}$ and observe that $\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+\frac{1}{n})^n}$. Thus $\lim \frac{s_{n+1}}{s_n} = \frac{1}{e}$ and, by Theorem 12.2 and Theorem 10.7(ii), we have $\liminf (s_n)^{1/n} = \limsup (s_n)^{1/n} = \frac{1}{e}$. By Theorem 10.7(ii) again, we get $\lim (s_n)^{1/n} = \frac{1}{e}$.

The followings are needed in the proof of Exercise 12.5.

Exercise 5.4 Case I: $-\infty < \inf S$. This case is proved in Exercise 4.9. See page 313. Case II: $-\infty = \inf S$. Then S is not bounded below and therefore $-S$ is not bounded above. Thus $\sup(-S) = \infty$, and hence $-\sup(-S) = -\infty = \inf S$. See the definitions on page 27.

Exercise 9.9 (c) See page 317.

Exercise 11.8 (a)

$$\begin{aligned}
\liminf_{n\to\infty} s_n &\overset{[i]}{=} \lim_{N\to\infty} \inf\{s_n \mid n > N\} \\
&\overset{[ii]}{=} \lim_{N\to\infty} (-\sup\{-s_n \mid n > N\}) \\
&\overset{[iii]}{=} -\lim_{N\to\infty} \sup\{-s_n \mid n > N\} \\
&\overset{[iv]}{=} -\limsup_{n\to\infty}(-s_n).
\end{aligned}$$

[i] & [iv] Definition 10.6; [ii] Exercise 5.4; [iii] Theorem 9.2 & Exercise 9.10(b).

Exercise 12.4 For $n > N$, we have

$$s_n \leq \sup\{s_n \mid n > N\} \quad \text{and} \quad t_n \leq \sup\{t_n \mid n > N\},$$

hence

$$s_n + t_n \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}.$$

Thus $\sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$ is an upper bound of $\{s_n + t_n \mid n > N\}$. Therefore, $\sup\{s_n + t_n \mid n > N\} \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$. Now apply Exercise 9.9(c) and Theorem 9.3.