Mat127a Discussion Section 8

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Exercise 17.2 (a) The domain of the following functions is $\mathbb{R}$.

$$(f+g)(x) = \begin{cases} x^2 & \text{for} \quad x < 0 \\ x^2 + 4 & \text{for} \quad x \geq 0. \end{cases}$$

$$(fg)(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x^2 & \text{for} \quad x \geq 0. \end{cases}$$

$$(f \circ g)(x) = 4 \quad \text{for} \quad x \in \mathbb{R}.$$

$$(g \circ f)(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 16 & \text{for} \quad x \geq 0. \end{cases}$$

(b) Only ␣ and ␣ are continuous.

Exercise 17.3 See page 323.

Exercise 17.6 This is clear from Exercise 17.5 and Theorem ␣.

Exercise 17.8 (a) Let $a, b \in \mathbb{R}$. If $a \geq b$, then

$$\begin{aligned} \frac{1}{2}(a+b) - \frac{1}{2}|a-b| &= \frac{1}{2}(a+b) - \frac{1}{2}(a-b) \\ &= b \\ &= \min\{a, b\}. \end{aligned}$$

If $a < b$, then

$$\begin{aligned} \frac{1}{2}(a+b) - \frac{1}{2}|a-b| &= \frac{1}{2}(a+b) - \frac{1}{2}(b-a) \\ &= a \\ &= \min\{a, b\}. \end{aligned}$$

Therefore, for any real numbers a and b, we have

$$\min\{a, b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|.$$

Now let $x \in \text{dom}(f) \cap \text{dom}(g)$. Because f and g are real-valued functions, we have $f(x), g(x) \in \mathbb{R}$ and

$$\min\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)|,$$

that is,

$$\Big(\min(f, g)\Big)(x) = \left(\frac{1}{2}(f+g) - \frac{1}{2}|f-g|\right)(x).$$

Therefore, we have that

$$\min(f, g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|.$$

(b) Let $a, b \in \mathbb{R}$. Consider the separate cases $a \geq b$ and $a < b$ to show that

$$\min\{a, b\} = -(\max\{-a, -b\}),$$

and then, as in (a), prove

$$\min(f, g) = -\max(-f, -g)$$

for any real-valued functions f and g.

(c) Use (b), Example 5, and Theorem 17.4(ii).

Exercise 17.12 (a) Let $q \in (a, b)$ be an irrational number. From Example 3 in page 65, we see that there is a sequence (r_n) of rational numbers in (a, b) such that $\lim r_n = q$. Now use the continuity of f to show $f(q) = 0$.

(b) Consider the function $h = f - g$ and use (a).

Exercise 17.14 (i) Let $x_0 \in \mathbb{Q}$. We write x_0 as $\frac{p}{q}$, where p, q are integers with no common factors and $q > 0$. By definition, $f(x_0) = \frac{1}{q} > 0$. Now consider a sequence (y_n) of irrational numbers such that $\lim y_n = x_0$.

(ii) Let $y_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let $\epsilon > 0$ be given. If $\epsilon > 1$, let δ be any positive number. Then $|y - y_0| < \delta$ implies

$$|f(y) - f(y_0)| = \begin{cases} |0 - 0| < \epsilon & \text{if} \quad y \in \mathbb{R} \setminus \mathbb{Q} \\ |\frac{1}{n} - 0| \leq 1 < \epsilon & \text{if} \quad y = \frac{m}{n} \in \mathbb{Q} \end{cases}$$

and we are done.

So suppose $\epsilon < 1$ from now on. Denote $N = \lfloor \frac{1}{\epsilon} \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Consider the set

$$A = \left\{ \frac{m}{n} \,\middle|\, m \in \mathbb{Z}, n \in \mathbb{N}, n \leq N \right\},$$

which has the property that

$$|a - b| > \frac{1}{N^2} \quad \text{for} \quad a, b \in A,\ a \neq b.$$

(This is because $\left| \frac{m_1}{n_1} - \frac{m_2}{n_2} \right| = \frac{|m_1 n_2 - m_2 n_1|}{n_1 n_2} > \frac{1}{N^2}$.) We now choose

$$\delta = \inf\{\, |y_0 - a| \mid a \in A\}.$$

Note that δ depends only on y_0 and ϵ, and $\delta > 0$ because of the above property of A. Observe that if $y_1 = \frac{m}{n} \in \mathbb{Q}$ satisfies $|y_1 - y_0| < \delta$, then by our choice of δ we see that $y_1 \notin A$, i.e. $n > N = \lfloor \frac{1}{\epsilon} \rfloor$, and hence $n > \frac{1}{\epsilon}$ and $f(y_1) = \frac{1}{n} < \epsilon$. Therefore, for any y with $|y - y_0| < \delta$, we have

$$|f(y) - f(y_0)| = \begin{cases} |0 - 0| < \epsilon & \text{if} \quad y \in \mathbb{R} \setminus \mathbb{Q} \\ |\frac{1}{n} - 0| < \epsilon & \text{if} \quad y = \frac{m}{n} \in \mathbb{Q}. \end{cases}$$

We are done with this case, too.