Mat127a Discussion Section 9

Prepared by Jaejeong Lee

Exercise 18.1 Note that if $-f(x_0) \geq -f(x)$ for all $x \in [a,b]$, then $f(x) \geq f(x_0)$ for all $x \in [a,b]$.

Exercise 18.2 If we replace $[a,b]$ with (a,b), then we cannon make the following two assertions any longer. Why? Explain.

"The number x_0 also must belong to the closed interval $[a,b]$."

"By the Bolzano-Weierstrass theorem, there is a subsequence (y_{n_k}) of (y_n) converging to a limit y_0 in $[a,b]$."

Exercise 18.3 Because $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, f has its local maximum $f(1) = 5$ and local minimum $f(3) = 1$. Also, we have $f(0) = 1$ and $f(5) = 21$. What are the (global) maximum and minimum of f on $[0,5)$? Drawing the graph of f would help.

Exercise 18.6 Use Exercise 18.5(a) by setting $f(x) = \cos x$ and $g(x) = x$. Note also that $0 \neq \cos 0$ and $\frac{\pi}{2} \neq \cos\frac{\pi}{2}$. For the proof of Exercise 18.5(a), see page 325.

Exercise 18.9 See page 325.

Exercise 19.1 See page 325.

(d) For each $\delta > 0$, we have the following:

$$\begin{aligned}\left| f\left(\frac{1}{\delta} + \frac{\delta}{2}\right) - f\left(\frac{1}{\delta}\right) \right| &= \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^3 - \left(\frac{1}{\delta}\right)^3 \\ &= \frac{3}{2\delta} + \frac{3\delta}{4} + \frac{\delta^3}{8} \\ &\geq 2\sqrt{\frac{3}{2\delta} \cdot \frac{3\delta}{4}} + \frac{\delta^3}{8} \\ &> \frac{3}{\sqrt{2}}\end{aligned}$$

Use this to show that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

Exercise 19.2 (a) Observe that $|f(x) - f(y)| = 3|x-y|$.

(b) Note that $|x^2 - y^2| = |x+y||x-y|$ and $|x+y| \leq 6$ for $x, y \in [0,3]$.

(c) Note that $\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy}$ and $\frac{1}{xy} \leq 4$ for $x, y \in [\frac{1}{2}, \infty)$.

Exercise 19.6 (a) $f'(x) = \frac{1}{2\sqrt{x}}$ is clearly unbounded on $(0,1]$. Because f is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$ by Theorem 19.2. Therefore, f is uniformly continuous on $(0,1]$.

(b) f' is bounded on $(1,\infty)$ since $0 < \frac{1}{2\sqrt{x}} < \frac{1}{2}$ for $x \in (1,\infty)$. By Theorem 19.6, f is uniformly continuous on $[1,\infty)$.

Exercise 19.8 (a) Given $x < y$, the Mean Value theorem says that there exists $c \in (x,y)$ such that

$$\frac{\sin x - \sin y}{x - y} = (\sin)'(c) = \cos c.$$

Therefore, we have $\left|\frac{\sin x - \sin y}{x-y}\right| = |\cos c| \leq 1$, that is,

$$|\sin x - \sin y| \leq |x - y|.$$

(b) Given $\epsilon > 0$, we choose $\delta = \epsilon$. Then $x, y \in \mathbb{R}$ and $|x-y| < \delta$ implies that

$$|\sin x - \sin y| \leq |x-y| < \delta = \epsilon.$$

Therefore, $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

Exercise 19.9 (a) See Exercises 17.3(f) and 17.9(c).

(b) Let S be a bounded subset of $\mathbb{R}$. Then we have $S \subset [-M, M]$ for some $M > 0$. By (a) f is continuous on $[-M,M]$, hence uniformly continuous on $[-M,M]$ by Theorem 19.2. Therefore, f is uniformly continuous on S.

(c) See page 326.