

Mat127a Homework Solutions 2

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Exercise 7.1 (a) $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16}$ (b) $\frac{4}{3}, 1, \frac{10}{11}, \frac{13}{15}, \frac{16}{19}$ (c) $\frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{4}{81}, \frac{5}{243}$ (d) $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$

Exercise 7.2 (a) converges to 0; (b) converges to $\frac{3}{4}$; (c) converges to 0; (d) diverges.

Exercise 7.4 (a) $\lim \frac{\sqrt{2}}{n} = 0$. (b) $\lim \left(1 + \frac{1}{n}\right)^n = e$.

Exercise 7.5

(a) $\lim(\sqrt{n^2+1}-n) = \lim \frac{1}{\sqrt{n^2+1}+n} = 0$.

(b) $\lim(\sqrt{n^2+n}-n) = \lim \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{2}$.

(c) $\lim(\sqrt{4n^2+n}-2n) = \lim \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{4}$.

Exercise 8.2 (a) $\lim \frac{n}{n^2+1} = 0$. For any given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$. Then $n > N = \frac{1}{\epsilon}$ implies $\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon$, as desired.

(b) $\lim \frac{7n-19}{3n+7} = \frac{7}{3}$. Given $\epsilon > 0$, let $N = \frac{106}{9\epsilon} - \frac{7}{3}$. If $n > N = \frac{106}{9\epsilon} - \frac{7}{3}$, then $n + \frac{7}{3} > \frac{106}{9\epsilon}$, hence $\frac{106}{3(3n+7)} < \epsilon$.

Therefore we have $\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| = \frac{106}{3(3n+7)} < \epsilon$.

(c) $\lim \frac{4n+3}{7n-5} = \frac{4}{7}$. If we are given an error $\epsilon > 0$, take $N = \frac{41}{49\epsilon} + \frac{5}{7}$. Then $n > N = \frac{41}{49\epsilon} + \frac{5}{7}$ implies $n - \frac{5}{7} > \frac{41}{49\epsilon}$, hence $\frac{41}{7(7n-5)} < \epsilon$, and hence $\left| \frac{4n+3}{7n-5} - \frac{4}{7} \right| = \frac{41}{7(7n-5)} < \epsilon$.

(d) $\lim \frac{2n+4}{5n+2} = \frac{2}{5}$. For $\epsilon > 0$, if we let $N = \frac{16}{25\epsilon} - \frac{2}{5}$, then $n > N = \frac{16}{25\epsilon} - \frac{2}{5}$ implies $n + \frac{2}{5} > \frac{16}{25\epsilon}$, hence $\frac{16}{5(5n+2)} < \epsilon$. Therefore we get $\left| \frac{2n+4}{5n+2} - \frac{2}{5} \right| = \frac{16}{5(5n+2)} < \epsilon$.

(e) $\lim \frac{1}{n} \sin n = 0$. Given an error $\epsilon > 0$, let $N = \frac{1}{\epsilon}$. Then $n > N = \frac{1}{\epsilon}$ implies $\left| \frac{1}{n} \sin n - 0 \right| = \frac{|\sin n|}{n} \leq \frac{1}{n} < \epsilon$, as desired.

Exercise 8.7 See page 315.

Exercise 8.9 Denote $s = \lim s_n$. (a) Suppose, on the contrary, that $s < a$. Let $\epsilon = a - s > 0$. Then there exists a number N such that $n > N$ implies $|s_n - s| < \epsilon = a - s$, which, by Exercise 3.7(b), implies

$$s_n < s + (a - s) = a.$$

Thus $s_n < a$ for infinitely many n ; a contradiction. Therefore, we must have $s \geq a$. (See page 316 for a slightly more complicated argument.)

(b) Similar to (a).

(c) This clearly follows from (a) and (b).

Exercise 9.1 In the followings, the logical order in which one apply theorems is

$$9.7(a) \rightarrow 9.2 \rightarrow 9.3 \rightarrow 9.6.$$

(a) $\lim \frac{n+1}{n} = \lim \left(1 + \frac{1}{n}\right) \stackrel{[i]}{=} \lim 1 + \lim \frac{1}{n} \stackrel{[ii]}{=} 1 + 0 = 1$.
[i] Theorem 9.3, [ii] Theorem 9.7(a).

(b) $\lim \frac{3n+7}{6n-5} = \lim \frac{3+\frac{7}{n}}{6-\frac{5}{n}} \stackrel{[i]}{=} \frac{\lim 3 + 7 \lim \frac{1}{n}}{\lim 6 - 5 \lim \frac{1}{n}} \stackrel{[ii]}{=} \frac{3+7 \cdot 0}{6-5 \cdot 0} = \frac{1}{2}$.
[i] Theorem 9.2 & 9.3 & 9.6, [ii] Theorem 9.7(a).

(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \lim \frac{17+\frac{73}{n}-\frac{18}{n^3}+\frac{3}{n^5}}{23+\frac{13}{n^2}} \stackrel{[i]}{=} \frac{\lim 17+73 \lim \frac{1}{n}-18 \lim \frac{1}{n^3}+3 \lim \frac{1}{n^5}}{\lim 23+13 \lim \frac{1}{n^2}} \stackrel{[ii]}{=} \frac{17+73 \cdot 0-18 \cdot 0+3 \cdot 0}{23+13 \cdot 0} = \frac{17}{23}$. [i] Theorem 9.2 & 9.3 & 9.6, [ii] Theorem 9.7(a).

Exercise 9.6 (a) If $a = \lim x_n$, then we must have

$$a = \lim x_n = \lim x_{n+1} = \lim 3x_n^2 \stackrel{[i]}{=} 3(\lim x_n)^2 = 3a^2,$$

where the equality [i] follows from Theorem 9.2 & 9.4. Thus $a = \frac{1}{3}$ or $a = 0$.

(b) We claim that $x_n = 3^{2^{n-1}-1}$ for $n \geq 1$. Then it clearly follows that $\lim x_n = +\infty$. To prove the claim, let the above sequence of assertions be (P_n) and we proceed by mathematical induction. The basis for induction P_1 is clearly true, since $3^{2^{1-1}-1} = 3^0 = 1 = x_1$. For the induction step, suppose that P_n is true. We then have

$$x_{n+1} = 3x_n^2 = 3(3^{2^{n-1}-1})^2 = 3^{1+2(2^{n-1}-1)} = 3^{2^n-1},$$

so P_{n+1} is also true and the induction step holds. Therefore, P_n is true for all $n \geq 1$.

(c) The moral here is that we cannot apply Limit Theorems 9.2-9.6 without verifying (i) whether the given sequence converges or not, and (ii) if converges, whether the limit is $\pm\infty$ or not.

Exercise 9.15 Let $x_n = \frac{a^n}{n!}$. Then $\frac{x_{n+1}}{x_n} = \frac{a}{n+1}$, so we have $\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \frac{|a|}{n+1} = 0$. From Exercise 9.12(a), we conclude $\lim x_n = 0$.

The followings are needed in the proof of Exercise 9.15.

Exercise 9.12 (a) If $L < 1$, select a real number a so that $L < a < 1$. Because $\lim \left| \frac{s_{n+1}}{s_n} \right| = L$, there exists a natural number N such that $n \geq N$ implies $\left| \left| \frac{s_{n+1}}{s_n} \right| - L \right| < a - L$, which, by Exercise 3.7(b), implies $\left| \frac{s_{n+1}}{s_n} \right| < L + (a - L) = a$, that is, $|s_{n+1}| < a|s_n|$. Therefore, for $n \geq N$, we have

$$\begin{aligned} |s_n| &< a|s_{n-1}| < a^2|s_{n-2}| < \\ &\dots < a^{n-N+1}|s_{N+1}| < a^{n-N}|s_N|. \end{aligned}$$

(To be rigorous, one has to show this using mathematical induction. *cf.* Exercise 1.8.) That is to say, we have $|s_n| < a^{n-N}|s_N|$ for all but finitely many n . We know $\lim a^{n-N} = 0$ from Theorem 9.7(b), hence $\lim a^{n-N}|s_N| = 0$ by Theorem 9.2. Therefore, in view of Exercise 8.5(b), we have $\lim |s_n| = 0$. And finally, by Exercise 8.6(a), we have $\lim s_n = 0$ as desired.

Exercise 8.5 See page 315.

Exercise 8.6 (a) This is clear in view of

$$|s_n - 0| = ||s_n| - 0|.$$