Mat127a Homework Solutions 5

Prepared by Jaejeong Lee

(1) Suppose that (X, τ) is a topological space and Y is a subset of X. Show that a subset D of Y is closed in the subspace topology if and only if D is the intersection of Y with some closed set C of X.

Solution $D = Y \cap C$ for some closed subset C of $X \iff Y - D = Y \cap (X - C)$ and $X - C$ is an open subset of $X \iff Y - D$ is open in $Y \iff D$ is closed in Y.

(2) List 12 (non-symmetric) topologies on a set with 4 elements.

Solution Let $X = \{a, b, c, d\}$. The followings are some 12 (non-symmetric) topologies on X.

$\tau_1 = \{\emptyset, X\}$.
$\tau_2 = \{\emptyset, X, \{a\}\}$.
$\tau_3 = \{\emptyset, X, \{a, b\}\}$.
$\tau_4 = \{\emptyset, X, \{a, b, c\}\}$.
$\tau_5 = \{\emptyset, X, \{a\}, \{a, b\}\}$.
$\tau_6 = \{\emptyset, X, \{a\}, \{a, b, c\}\}$.
$\tau_7 = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$.
$\tau_8 = \{\emptyset, X, \{a\}, \{b, c, d\}\}$.
$\tau_9 = \{\emptyset, X, \{a, b\}, \{c, d\}\}$.
$\tau_{10} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
$\tau_{11} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$.
$\tau_{12} = \{\emptyset, X, \{a\}, \{a, b\}, \{c, d\}\}$.

(3) Prove that the Cantor set is not connected.

Solution Let $F = \cap_{n=1}^{\infty} F_n$ be the Cantor set as defined on page 85. Let $U = F \cap \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $V = F \cap \left(\frac{1}{2}, \frac{3}{2}\right)$. Then U and V are nonempty open subsets of F, $F = U \cup V$, and $U \cap V = \emptyset$. Therefore, (U, V) is a separation of F.

(4) Prove that the Cantor set is compact.

Solution By Theorem 13.10, F is closed and bounded. By the Heine-Borel Theorem 13.12, F is compact.

Exercise 13.1 See page 320.

Exercise 13.3 (a) It is clear that d satisfies D1 and D2 of Definition 13.1. For D3, let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in B$. Then for any $j \in \mathbb{N}$ we have

$$|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j| \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

Therefore, we get $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

(b) See page 320.

Exercise 13.11 See page 321.

Exercise 13.12 (a) To show that E is compact, let $\mathcal{U}$ be a covering of E by open sets in S. Since E is closed in F, $E = \tilde{E} \cap F$ for some closed subset $\tilde{E}$ of S. Then $\mathcal{V} = \mathcal{U} \cup \{S - \tilde{E}\}$ is an open cover for F. Because F is compact, some finite subcover $\mathcal{V}'$ of $\mathcal{V}$ covers F. Now $\mathcal{V}' - \{S - \tilde{E}\}$ is a finite subcover of $\mathcal{U}$ for E.

(b) Let $C = \cup_{i=1}^{k} C_i$ be a finite union of compact sets in S. Let $\mathcal{U}$ be an open covering for C. Then $\mathcal{U}$ is also an open covering for C_i for all $i = 1, \ldots, k$. For each i, since C_i is compact, some finite subcover $\mathcal{U}_i$ of $\mathcal{U}$ covers C_i. Now $\cup_{i=1}^{k} \mathcal{U}_i$ is a finite subcover of $\mathcal{U}$ for C.