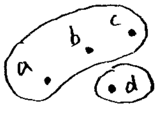
Discussion 3

$\boxed{5.6.5}$ $= \langle x, y \mid x^3 = 1,\ y^2 = 1,\ yx = x^{-1}y \rangle = \{1, x, x^2, y, xy, x^2y\}$

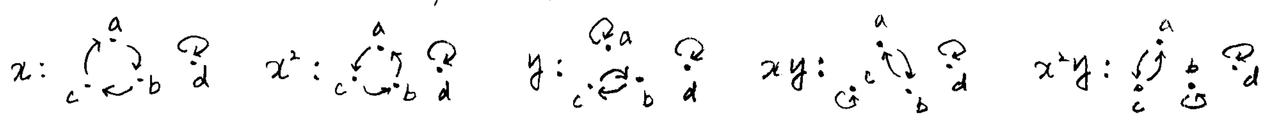
Let $G = S_3$ operate on $S = \{a, b, c, d\}$. By Proposition (6.4), we have for any element of S, say a, $|\mathrm{Orb}(a)| = |G/G_a| = |G|/|G_a| =$ divisor of $|G| = 1, 2, 3,$ or 6. Because $|\mathrm{Orb}(a)| \le |S| = 4$, $|\mathrm{Orb}(a)| = 1, 2,$ or 3. So, possible formulae representing the decomposition of S into orbits are $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$.

i. $3+1$) Without loss of generality, we may assume $\{a, b, c\} = \mathrm{Orb}(a) = \mathrm{Orb}(b) = \mathrm{Orb}(c)$ and $\{d\} = \mathrm{Orb}(d)$. By Proposition (6.4), we see $|G_a| = 6/3 = 2$, $|G_d| = 6/1 = 6$, so we may set $G_a = \{1, y\}$ and $G_d = G$. Then we have the following correspondences:

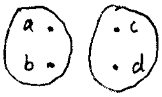


$\{a, b, c\} \leftrightarrow \{\{1, y\}, x\{1, y\}, x^2\{1, y\}\}$ and $\{d\} \leftrightarrow \{G\}$.

Therefore, G fixes $\{d\}$ and operates on $\{a, b, c\}$ in the standard manner.



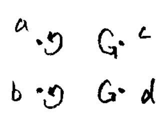
ii. $2+2$) We may set $\{a, b\} = \mathrm{Orb}(a) = \mathrm{Orb}(b)$ and $\{c, d\} = \mathrm{Orb}(c) = \mathrm{Orb}(d)$.

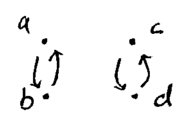


Since $|G_a| = |G_b| = |G_c| = |G_d| = 6/2 = 3$ and $\{1, x, x^2\}$ is the only subgroup of order 3, we have $G_a = G_b = G_c = G_d = \{1, x, x^2\}$, i.e., $\{1, x, x^2\}$ fixes $S = \{a, b, c, d\}$. From the correspondence:

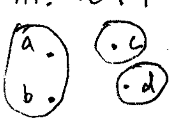
$\{a, b\} \leftrightarrow \{\{1, x, x^2\}, y\{1, x, x^2\}\} \leftrightarrow \{c, d\}$,

we see each element of $\{y, xy, x^2y\}$ acts on $\{a, b\}$ and $\{c, d\}$ as a transposition.

$1, x, x^2$: 

y, xy, x^2y : 

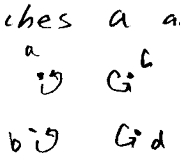
iii. $2+1+1$) Let $\{a, b\} = \mathrm{Orb}(a) = \mathrm{Orb}(b)$, $\{c\} = \mathrm{Orb}(c)$, and $\{d\} = \mathrm{Orb}(d)$.

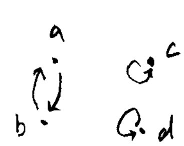


As in ii), $G_a = G_b = \{1, x, x^2\}$ and $G_c = G_d = G$. We have:

$\{a, b\} \leftrightarrow \{\{1, x, x^2\}, y\{1, x, x^2\}\}$, $\{c\} \leftrightarrow \{G\} \leftrightarrow \{d\}$.

Therefore, $\{1, x, x^2\}$ fixes S and each element of $\{y, xy, x^2y\}$ fixes $\{c, d\}$ and switches a and b.

$1, x, x^2$: 

y, xy, x^2y : 

iv. $1+1+1+1$) Each element of S is fixed by G, i.e., G operates on S trivially.

G : $\circlearrowleft a$ $\quad c \circlearrowleft$
$\quad\;\; \circlearrowleft b$ $\quad d \circlearrowleft$

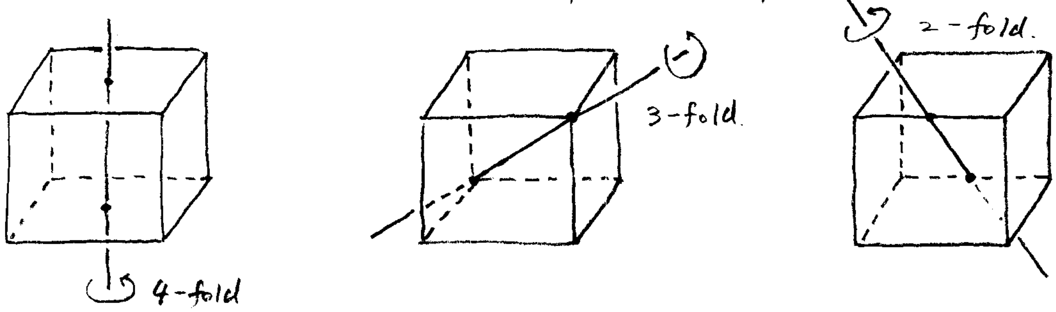
In fact, if we use Proposition (8.2), then the problem is equivalent to finding all homomorphism $\varphi : S_3 \longrightarrow S_4$. Since x and y generate S_3, it amounts to finding $\varphi(x)$ and $\varphi(y)$ with $\varphi(x)\varphi(y) = \varphi(x)^{-1}\varphi(y)$. By $\boxed{2.4.16}$ (p72), the possible images of x and y are:

$$\varphi(x) = \begin{cases} 1 & ① \\ (123), (132) & ② \end{cases} \qquad \varphi(y) = \begin{cases} 1 & ⓐ \\ (12), (23), (13) & ⓑ \\ (12)(34), (13)(24), (14)(23) & ⓒ \end{cases}$$

(Here, WLOG, we let $4 \in \{1, 2, 3, 4\}$ be special). Verify that combinations ②-ⓐ and ②-ⓒ do not satisfy $\varphi(x)\varphi(y) = \varphi(x)^{-1}\varphi(y)$ and the rest four correspond to: $3+1$ (2-b), $2+2$ (1-c), $2+1+1$ (1-b), $1+1+1+1$ (1-a).

$\boxed{5.7.4}$

Let $G = Rot(C)$ be the group of rotational symmetries of a cube C. The cube C has three 4-fold axes (opposite faces), four 3-fold axes (four diagonals), and six 2-fold axes (six pairs of opposite edges).

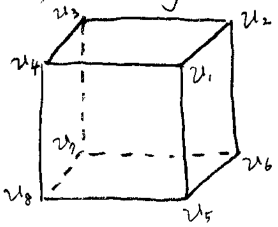


So $|G| = 3 \times (4-1) + 4 \times (3-1) + 6 \times (2-1) + 1 = 24$. Let's verify this using Proposition (7.2). Since $|S_v| = 8$, $|S_e| = 12$, $|S_f| = 6$, and $|H_v| = 3$, $|H_e| = 2$, $|H_f| = 4$, we see

$$\begin{aligned} |G| &= |H_v||S_v| = 3 \times 8 = 24 \\ &= |H_e||S_e| = 2 \times 12 = 24 \\ &= |H_f||S_f| = 4 \times 6 = 24. \end{aligned}$$

(Note G operates on S_v, S_e, S_f transitively.)

Let's now determine formulas like (7.3) (cf. Example (7.4)). We make the following convention.



$$S_v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$
$$S_e = \{e_{12}, e_{34}, e_{56}, e_{78}, e_{14}, e_{23}, e_{58}, e_{67}, e_{15}, e_{26}, e_{37}, e_{48}\}$$
$$S_f = \{f_{1234}, f_{5678}, f_{1256}, f_{3478}, f_{1458}, f_{2367}\}$$

i) $|S_v| = 8 = 1 + 1 + 3 + 3$ (H_{v_1} acts on S_v)

$\{v_1\}, \{v_7\}, \{v_2, v_4, v_5\}, \{v_3, v_8, v_6\}$

$= 2 + 2 + 2 + 2$ ($H_{e_{12}}$ acts on S_v)

$\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}$

$= 4 + 4$ ($H_{f_{1234}}$ acts on S_v)

$\{v_1, v_2, v_3, v_4\}, \{v_5, v_6, v_7, v_8\}$.

ii) $|S_e| = 12 = 3 + 3 + 3 + 3$ (H_{v_1} acts on S_e)

$\{e_{12}, e_{14}, e_{15}\}, \{e_{23}, e_{48}, e_{56}\}, \{e_{34}, e_{58}, e_{26}\}, \{e_{37}, e_{78}, e_{67}\}$

$= 1 + 1 + 2 + 2 + 2 + 2 + 2$ ($H_{e_{12}}$ acts on S_e)

$\{e_{12}\}, \{e_{78}\}, \{e_{23}, e_{15}\}, \{e_{14}, e_{26}\}, \{e_{34}, e_{56}\}, \{e_{37}, e_{58}\}, \{e_{48}, e_{67}\}$.

$= 4 + 4 + 4$ ($H_{f_{1234}}$ acts on S_e)

$\{e_{12}, e_{23}, e_{34}, e_{41}\}, \{e_{56}, e_{67}, e_{78}, e_{85}\}, \{e_{15}, e_{26}, e_{37}, e_{48}\}$

iii) $|S_f| = 6 = 3 + 3$ (H_{v_1} acts on S_f)

$\{f_{1234}, f_{1458}, f_{1256}\}, \{f_{2367}, f_{3478}, f_{5678}\}$.

$= 2 + 2 + 2$ ($H_{e_{12}}$ acts on S_f)

$\{f_{1234}, f_{1256}\}, \{f_{2367}, f_{1458}\}, \{f_{5678}, f_{3478}\}$.

$= 1 + 1 + 4$ ($H_{f_{1234}}$ acts on S_f)

$\{f_{1234}\}, \{f_{5678}\}, \{f_{1256}, f_{2367}, f_{3478}, f_{1458}\}$.

<u>Exercise</u> Let $S_d = \{d_{17}, d_{28}, d_{35}, d_{46}\}$ be the set of diagonals. Show each element of $G = Rot(C)$ corresponds to a permutation on S_d and this assignment $G \longrightarrow Perm(S_d) \cong S_4$ is ~~one-to-one~~ onto. Since $|G| = |S_4| = 24$, we have $G \cong S_4$.