# Discussion 4

## 5. 5. 10

**Review of Proposition 2.8, (p114)**

$V$, $W$ : vector spaces, $\dim V = n$, $\dim W = m$.

$\mathcal{B}$, $\mathcal{B}'$ : ordered bases for $V$, $\quad \mathcal{C}$, $\mathcal{C}'$ : ordered bases for $W$.

$T : V \longrightarrow W$, a linear transformation

$[T]_{\mathcal{B}}^{\mathcal{C}}$ : matrix representation of $T$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$.

$Q = [Id_V]_{\mathcal{B}}^{\mathcal{B}'}$, $\quad P = [Id_W]_{\mathcal{C}}^{\mathcal{C}'}$ : matrices of change of basis.

$$\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "T"] \arrow[d, "Id_V"'] & W_{\mathcal{C}} \arrow[d, "Id_W"] \\ V_{\mathcal{B}'} \arrow[r, "T"'] & W_{\mathcal{C}'} \end{tikzcd} \quad \overset{\text{Prop (2.8)}}{\Longrightarrow} \quad \begin{tikzcd} \mathbb{R}^n \arrow[r, "{[T]_{\mathcal{B}}^{\mathcal{C}}}"] \arrow[d, "{Q = [Id_V]_{\mathcal{B}}^{\mathcal{B}'}}"'] & \mathbb{R}^m \arrow[d, "{[Id_W]_{\mathcal{C}}^{\mathcal{C}'} = P}"] \\ \mathbb{R}^n \arrow[r, "{[T]_{\mathcal{B}'}^{\mathcal{C}'}}"'] & \mathbb{R}^m \end{tikzcd}$$

$$T = Id_W \circ T \circ (Id_V)^{-1} \Longrightarrow [T]_{\mathcal{B}'}^{\mathcal{C}'} = [Id_W]_{\mathcal{C}}^{\mathcal{C}'} [T]_{\mathcal{B}}^{\mathcal{C}} [Id_V]_{\mathcal{B}'}^{\mathcal{B}}$$
$$= [Id_W]_{\mathcal{C}}^{\mathcal{C}'} [T]_{\mathcal{B}}^{\mathcal{C}} \left([Id_V]_{\mathcal{B}}^{\mathcal{B}'}\right)^{-1}$$
$$= P [T]_{\mathcal{B}}^{\mathcal{C}} Q^{-1}.$$

(a) We need to verify (5.1) (p176). Clearly, $(I, I)(A) = I A I^{-1} = A$. Since $(P_2, Q_2)(P_1, Q_1)(A) = (P_2, Q_2)(P_1 A Q_1^{-1}) = P_2 (P_1 A Q_1^{-1}) Q_2^{-1} = (P_2 P_1) A (Q_2 Q_1)^{-1} = (P_2 P_1, Q_2 Q_1)(A)$, and $(P_2, Q_2)(P_1, Q_1) = (P_2 P_1, Q_2 Q_1)$ in $G$, the associative law holds, too.

( Exercise What if we defined $(P, Q)(A) = PAQ$ ? ).

(b). The operation defined in (a) is just change of bases. Proposition 2.9 (p114) saids that if $\operatorname{rank} A = r$, then $A$ is in the orbit of the matrix of the form (2.10) (p114), that is, matrices of the same rank are in the same $G$-orbit. So, if $k = \min(m, n)$, then

$S = \{$ real $m \times n$ matrices $\}$
$= \{0\} \cup \{$ matrices of rank $= 1\} \cup \{$ matrices of rank $= 2\} \cup \dots \cup \{$ matrices of rank $= k\}$.

(c). Let $P \in GL_m(\mathbb{R})$ and $Q^{-1} = A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \in GL_n(\mathbb{R})$, where $A_{11} \in GL_m(\mathbb{R})$. Suppose $(P, Q) \in \operatorname{Stab}([I|0])$, i.e., $P[I|0]Q^{-1} = [I|0]$.

$$[I|0] = P[I|0]Q^{-1} = P[I|0]\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22}\end{array}\right) = [P|0]\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22}\end{array}\right)$$

$$= [PA_{11} \mid PA_{12}].$$

So we have $PA_{11} = I$ and $PA_{12} = 0$, i.e., $A_{11} = P^{-1}$ and $A_{12} = P^{-1}0 = 0$.

Thus, $\mathrm{Stab}([I|0]) = \left\{ \left( P, \left(\begin{array}{c|c} P^{-1} & 0 \\ \hline * & \# \end{array}\right)\right) \,\middle|\, P \in GL_m(\mathbb{R}) \right\}$

$\boxed{5.8.5}$

Without loss of generality, we may set $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, and $B = \{4, 5\}$. By $\boxed{5.8.4}$, the action corresponds to a homomorphism $\varphi : G \longrightarrow S_3 \times S_2 = S_3 \times C_2$. Because the action is faithful, the map $\varphi$ is injective. Thus, we may identify $G$ with its image $\varphi(G) \subset S_3 \times C_2$, i.e., we may regard $G$ as a subgroup of $S_3 \times C_2$. In particular, $|G| \le |S_3 \times C_2| = 12$. By the counting formula, $|G|$ is a multiple of $|A| = 3$ and $|B| = 2$. So we must have $|G| = 6$ or $|G| = 12$.

If $|G| = 12$, then $G \cong S_3 \times C_2$. $S_3$ acts on $A = \{1, 2, 3\}$ as usual, and the nontrivial element of $C_2$ transpose 4 and 5.

If $|G| = 6$, then $G \cong C_3 \times C_2$ or $G \cong S_3$, since these are the only groups of order 6.

i) $G \cong C_3 \times C_2$ : if $C_3 = \langle x \rangle$ and $C_2 = \langle y \rangle$, then the action is defined by sending $x$ to the permutation $(1\,2\,3)$ and $y$ to $(45)$.

ii) $G \cong S_3$ : Note $G$ is not the subgroup $S_3 \times \{1\}$ of $S_3 \times C_2$, because it would imply the orbits are $\{1, 2, 3\}$, $\{4\}$, $\{5\}$. Let $S_3 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2y \rangle = \{1, x, x^2, y, xy, x^2y\}$ and $C_2 = \langle a \mid a^2 = 1\rangle = \{1, a\}$. Let $X = (x, 1)$ and $Y = (y, a)$. One can verify $X^3 = Y^2 = (1, 1)$ and $YX = X^2Y$ in $S_3 \times C_2$. $G$ is the subgroup of $S_3 \times C_2$ generated by $X$ and $Y$, which must be isomorphic to $S_3$. $X$ corresponds to $(1\,2\,3)$, and $Y$, $XY$, $X^2Y$ correspond to $(45)$.

<u>Remark</u>. In the last case of $G = S_3$, we can define a homomorphism $S_3 \longrightarrow S_3 \times C_2$ directly by $\tau \longmapsto (\tau, \mathrm{sgn}(\tau))$.

$\boxed{6.1.5}$

As we observed in <u>Discussion 1</u>, conjugation operation does not change the properties of being orientation-preserving and having a fixed point. More precisely, for $f, g \in M$, we have

i) $f$ is orientation-preserving $\iff$ $gfg^{-1}$ is orientation-preserving

ii) $f$ fixes $x$ $\iff$ $gfg^{-1}$ fixes $g(x)$.

Because these two properties determine the type (rotation, translation, reflection, glide) of a motion, the conjugation operation does not change the types of motion. Moreover, because the motions are rigid, i.e., distance-preserving, the length or angle of motion is also invariant under conjugation. For example, a rotation by $\theta$ is conjugate to another rotation by $\pm\theta$ (possibly around different fixed point) and a translation by $\vec{a}$ is conjugate to some translation by $\vec{b}$ with $|\vec{b}| = |\vec{a}|$. To see this more precisely, we use (2.4) and (2.5) on page 159. Because the group $M$ is generated by $t_a$, $\rho_\theta$, and $r$, it suffices to investigate conjugations between them. (Why?). We have:

$$\left.\begin{aligned} t_b t_a t_b^{-1} &= t_a \\ \rho_\theta t_a \rho_\theta^{-1} &= t_{\rho_\theta(a)} \\ r t_a r^{-1} &= t_{r(a)} \end{aligned}\right\} = \text{translations and } |a| = |\rho_\theta(a)| = |r(a)|.$$

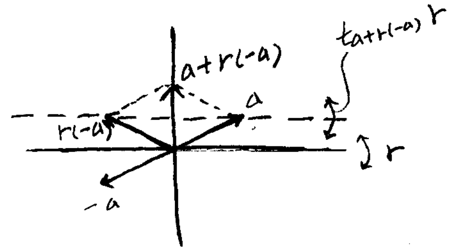
$$\left.\begin{aligned} t_a \rho_\theta t_a^{-1} &= t_{a+\rho_\theta(-a)} \rho_\theta \\ \rho_{\theta'} \rho_\theta \rho_{\theta'}^{-1} &= \rho_\theta \\ r \rho_\theta r^{-1} &= \rho_{-\theta} \end{aligned}\right\} = \text{rotations and } |\theta| = |-\theta|.$$

(cf. Corollary (2.8) for the first one)

cf. Proposition (2.11).

$$\left.\begin{aligned} t_a r t_a^{-1} &= t_{a+r(-a)} r \\ \rho_\theta r \rho_\theta^{-1} &= \rho_{2\theta} r \\ r r r^{-1} &= r \end{aligned}\right\} = \text{reflections (not glides).}$$

cf. $\boxed{5.2.7}$




Therefore, we see

$$M = \{\text{the identity}\} \cup \bigcup_{r>0} \{\text{translations by } \vec{a} \text{ and } |\vec{a}| = r\}$$
$$\cup \bigcup_{0<\theta\le\pi} \{\text{rotations by } \theta \text{ or } -\theta\}$$
$$\cup \bigcup_{r>0} \{\text{glides along } \vec{a} \text{ and } |\vec{a}| = r\}$$
$$\cup \{\text{reflections}\}.$$