Discussion 6

7.2.6

It's obviously true in view of the following fact: if $A \in \mathrm{Mat}(m \times n)$, $P \in GL_n(\mathbb{F})$, $Q \in GL_m(\mathbb{F})$, then $\mathrm{rank}(A) = \mathrm{rank}(QA) = \mathrm{rank}(AP) = \mathrm{rank}(QAP)$. One can also see it from Sylvester's law (p245): by Theorem 2.9 (b), there is a matrix $Q \in GL_n(\mathbb{F})$ such that $I_{p,m,z} = Q^t A Q$, where $I_{m,p,z}$ is of the form (2.10). Note $\mathrm{rank}(A) = p + m$. Since $A = P^t A' P$, we have $I_{p,m,z} = Q^t A Q = (PQ)^t A' (PQ)$, which implies $\mathrm{rank}(A') = p + m$, too.

7.2.7

Counterexample: Let $A = I_n$ be the matrix of $\langle , \rangle$ with respect to some basis. It clearly has n ones for eigenvalues. By Corollary 1.13 (p239), the matrix of $\langle , \rangle$ with respect to different basis is QAQ^t for some invertible Q. If $Q = 2I$, then $QAQ^t = 4I$, which has n fours for eigenvalues. Thus, the eigenvalues of a matrix representing $\langle , \rangle$ are not independent of basis. (cf. Proposition 4.17, p122).

7.2.23

(a) Note that this is a counterexample to Theorem 2.9 in case $\mathbb{F} \neq \mathbb{R}$.
Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_2)$, then $\det(P) = ad - bc = 1$. Now observe $PAP^t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2ab & ad+bc \\ ad+bc & 2ab \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ since $2 = 0$ and $ad + bc = ad - bc$ in $\mathbb{F}_2$. Thus, A is not diagonalizable as the matrix of a bilinear form.

(b) The action preserves rank and symmetry of matrices. So we have the following rough decomposition:

	rank = 0	rank = 1	rank = 2	
symmetric.	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
non-symm.	None	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	

By direct computation, one can check that this is the orbit decomposition if we further decompose the symmetric rank = 2 set into $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$. cf. (a). cf. (2.10) (p85).

7.3.10

Let $\mathbb{R}^n$ be endowed with the standard dot product $(\ ,\ )$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by left multiplication by A, i.e., $T(x) = Ax$.

(a) Let $x \in \ker T$ and $Ay \in \operatorname{im} T$. Since $Ax = 0$, we have $(x, Ay) = x^tAy = (x^tAy)^t = y^tA^tx = y^tAx = (y, Ax) = (y, 0) = 0$. Thus, $(\ker T) \perp (\operatorname{im} T)$ and $(\operatorname{im} T) \subset (\ker T)^\perp$. The standard dot product $(\ ,\ )$ on $\mathbb{R}^n$ is positive definite and so is its restriction to $\ker T$. In particular, this restriction of $(\ ,\ )$ to $\ker T$ is non-degenerate and by Proposition 2.7, we have $\mathbb{R}^n = (\ker T) \oplus (\ker T)^\perp$. So, $\dim(\ker T)^\perp = n - \dim(\ker T) = \dim(\operatorname{im} T)$, the first equality by Proposition 6.9 (p103) and the second by the dimension formula (p110). Since $(\operatorname{im} T) \subset (\ker T)^\perp$ and they are of the same dimension, we have $(\operatorname{im} T) = (\ker T)^\perp$ and $\mathbb{R}^n = (\ker T) \oplus (\operatorname{im} T)$.

(b). ($\Longrightarrow$) If $x = x_k + x_i$ is the unique expression of x with respect to the direct sum decomposition of $\mathbb{R}^n = (\ker T) \oplus (\operatorname{im} T)$, then T is defined by $T(x) = Ax = x_i$. Since $x_i = 0 + x_i$ is the expression of x_i, we have $T^2(x) = A^2x = Ax_i = x_i = Ax$. Since x is arbitrary, $A^2 = A$.

($\Longleftarrow$) Let $x = x_k + x_i$ be the expression of x with respect to the direct sum decomposition in (a), i.e., $x_k \in \ker T$ and $x_i = T(y) \in \operatorname{im} T$ for some y. If $A^2 = A$, i.e, $T^2 = T$, then $T(x) = T(x_k + x_i) = T(x_k) + T(x_i) = 0 + T^2y = Ty = x_i$. Thus T is an (orthogonal) projection onto $\operatorname{im} T$.