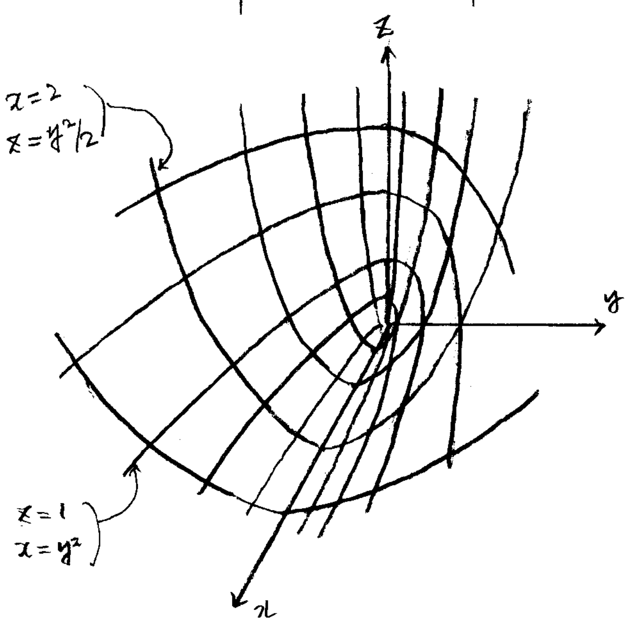
Discussion 7

$\boxed{8.1.4}$

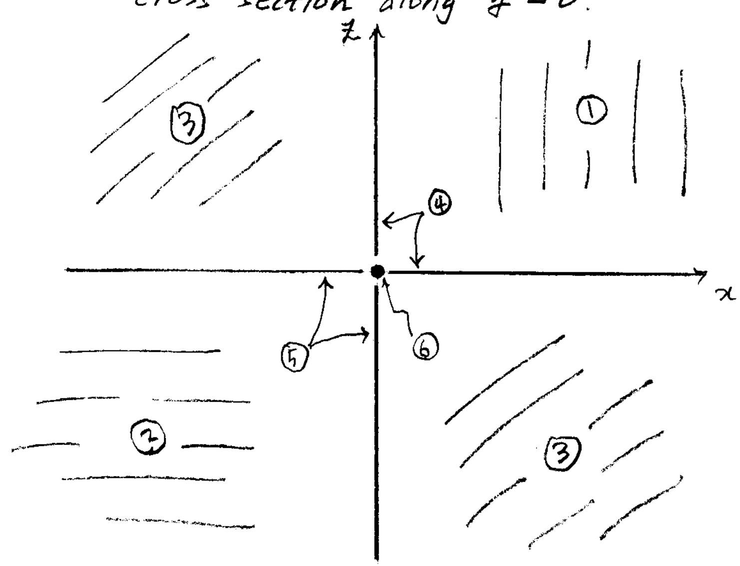
Note that, under ~~the~~ action $P, A \longrightarrow PAP^t$, the rank of A, the sign of $\det A$, and positive/negative definiteness of A (in case A has full rank) are invariant. Recall $\boxed{7.2.6}$, Theorem 1.25 (p 242), $\boxed{7.2.11}$. Let $A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. We have the following table. (Note $\det A = xz - y^2$)

rank $A = 2$	$\det A > 0$	$x > 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	①
		$x < 0$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	②
	$\det A < 0$		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	③
rank $A = 1$	$\det A = 0$	$y = 0,\ x > 0,\ z = 0$ $y = 0,\ x = 0,\ z > 0$ $y \neq 0,\ x > 0,\ z > 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	④
		$y = 0,\ x < 0,\ z = 0$ $y = 0,\ x = 0,\ z < 0$ $y \neq 0,\ x < 0,\ z < 0.$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	⑤
rank $A = 0$			$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	⑥



Half of the surface $xz - y^2 = 0$
$\Longleftrightarrow$ orbit of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Cross-section along $y = 0$.



#1 Given a hermitian/symmetric matrix, find an orthogonal basis.

i) $A = \begin{pmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 2 \end{pmatrix}$, (u_1, u_2, u_3) : the standard basis for $\mathbb{C}^3$.

$$w_1 = u_1 = (1, 0, 0)^t$$

$$w_2 = u_2 - \frac{\overline{\langle u_2, w_1 \rangle}}{\langle w_1, w_1 \rangle} w_1 = u_2 - \frac{\overline{(i)}}{(1)} w_1 = u_2 + i w_1 = (i, 1, 0)^t$$

$$w_3 = u_3 - \frac{\overline{\langle u_3, w_1 \rangle}}{\langle w_1, w_1 \rangle} w_1 - \frac{\overline{\langle u_3, w_2 \rangle}}{\langle w_2, w_2 \rangle} w_2 = u_3 - \frac{\overline{(0)}}{(1)} w_1 - \frac{\overline{(-i)}}{(1)} w_2 = (1, -i, 1)^t$$

$$P = \begin{pmatrix} 1 & i & 1 \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}, \quad P^* A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -i & 0 & 1 & 0 & 0 \\ i & 2 & i & 0 & 1 & 0 \\ 0 & -i & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & i & 0 \\ i & 1 & i & 0 & 1 & 0 \\ 0 & -i & 2 & 0 & 0 & 1 \end{array}\right) \dashrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & i & 0 \\ 0 & 1 & i & 0 & 1 & 0 \\ 0 & -i & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & i & 1 \\ 0 & 1 & 0 & 0 & 1 & -i \\ 0 & -i & 1 & 0 & 0 & 1 \end{array}\right) \dashrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & i & 1 \\ 0 & 1 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (P^* A P \mid P)$$

ii) $A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, (u_1, u_2, u_3) : the standard basis for $\mathbb{R}^3$.

$$w_1 = u_1 = (1, 0, 0)^t$$

$$w_2 = u_2 - \frac{\langle u_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = u_2 - \frac{(-1)}{(1)} w_1 = (1, 1, 0)^t$$

$$w_3 = u_3 - \frac{\langle u_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle u_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = u_3 - \frac{(3)}{(1)} w_1 - \frac{(4)}{(1)} w_2 = (-7, -4, 1)^t.$$

$$P = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^t A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -24 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \dashrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 3 & 4 & -8 & 0 & 0 & 1 \end{array}\right) \dashrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 4 & -8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -7 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 4 & -24 & 0 & 0 & 1 \end{array}\right)$$

$$\dashrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -7 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & -24 & 0 & 0 & 1 \end{array}\right) = (P^t A P \mid P).$$