Discussion 8

#1

Let $x = \begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Verify the following relations: $x^n = y^2 = -I$ and $yx = x^{-1}y$. Since $|x| =$ order of $x = 2n$, the elements $I, x, \cdots, x^{2n-1}$ are distinct. It follows that the elements $y, xy, \cdots, x^{2n-1}y$ are also distinct. Suppose $x^j = x^i y$ for some $0 \le i, j \le 2n-1$. Then $x^{j-i} = y$, which is impossible because x^{j-i} is diagonal but y is not. So the elements $I, x, \cdots, x^{2n-1}, y, xy, \cdots, x^{2n-1}y$ are distinct. Since ~~the~~ above relations can be used to reduce any product of x, x^{-1}, y, y^{-1} to the form $x^i y^j$, $0 \le i \le 2n-1$, $0 \le j \le 1$, we see

$$\begin{aligned} G_n &= \{ I, x, x^2, \cdots, x^{2n-1}, y, xy, x^2y, \cdots, x^{2n-1}y \} \\ &= \{ x^i y^j \mid 0 \le i < 2n, \ 0 \le j < 2 \} \\ &= \langle x, y \mid x^{2n} = I, \ x^n = y^2, \ yx = x^{-1}y \rangle \end{aligned}$$

and $|G_n| = 4n$.

Let $X = x\{\pm I\}$ and $Y = y\{\pm I\}$. Then the quotient group $G_n/\{\pm I\}$ is generated by X and Y. Since X and Y satisfy $X^n = Y^2 = \{\pm I\}$ and $YX = X^{-1}Y$, we have $G_n/\{\pm I\} \cong D_n$ by Proposition 3.6 (p165).

8.2.2

Since $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in SO_2 \subseteq SU_2 \subseteq U_2$ is unitary, it is normal and diagonalizable by Theorem 7.3 (p259). cf. $\boxed{4.6.2}$. Let's first find the eigenvalues of A: $0 = \det(\lambda I - A) = \lambda^2 - 2\cos\alpha\,\lambda + 1$, therefore $\lambda = \cos\alpha \pm \sqrt{\cos^2\alpha - 1} = \cos\alpha \pm i|\sin\alpha| = \cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}$. Since $\begin{pmatrix} i\sin\alpha & \sin\alpha \\ -\sin\alpha & i\sin\alpha \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -i\sin\alpha & \sin\alpha \\ -\sin\alpha & -i\sin\alpha \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the corresponding unit eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$. (Note $\langle \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \rangle = \overline{(i)}(i) + \overline{(1)}(1) = 2$ and similarly $\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle = 2$). By $\boxed{7.4.19}$, $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ are automatically orthogonal. $\left(\langle \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle = \overline{(i)}(1) + \overline{(1)}(i) = 0 \right)$. Set $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Then $P \in SU_2$ and $P^{-1}AP = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$. Therefore, we have

$$P^{-1}(SO_2)P = T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \,\middle|\, \lambda\bar{\lambda} = 1 \right\}$$

and the subgroup $SO_2 \subseteq SU_2$ is conjugate to $T \subseteq SU_2$.