Discussion 10

$\boxed{10.3.2}$

Let $I \subseteq R$ be an ideal of a ring R. Suppose I contains a unit, say, $a \in I$. Then there exists $b \in R$ such that $ba = 1$. Since I is an ideal, $ba \in I$, so $1 \in I$. Thus, $I \supseteq (1) = R$. Therefore, $I = R = (1)$.

$\boxed{10.3.3}$

$$
\begin{array}{r|rrrrr}
 & & & 1 & 2 & -2 \\
\hline
1\ 1\ 1 & 1 & 3 & 1 & 6 & 10 \\
 & 1 & 1 & 1 & & \\
\hline
 & & 2 & 0 & 6 & \\
 & & 2 & 2 & 2 & \\
\hline
 & & & -2 & 4 & 10 \\
 & & & -2 & -2 & -2 \\
\hline
 & & & & 6 & 12
\end{array}
$$

In $(\mathbb{Z}/n\mathbb{Z})[x] = \mathbb{Z}_n[x]$, we have

$$\bar{1}x^4 + \bar{3}x^3 + \bar{1}x^2 + \bar{6}x + \overline{10}$$
$$= (\bar{1}x^2 + \bar{2}x - \bar{2})(\bar{1}x^2 + \bar{1}x + \bar{1}) + (\bar{6}x + \overline{12})$$

So, the remainder $\bar{6}x + \overline{12}$ is zero iff $\bar{6} = \overline{12} = \bar{0}$ in $\mathbb{Z}_n$ iff $n \mid 6$ iff $n = 1, 2, 3, 6$.

$\boxed{10.3.18}$

Note that, in the ring $\mathbb{Z}$, we have $(n) = n\mathbb{Z}$ and $\mathbb{Z}/(n) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$.

(a) Define a group homomorphism $\varphi : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_5$ by $\varphi(\bar{1}) = (\bar{1}, \bar{1})$ (Since $\mathbb{Z}_{10}$ is a cyclic group generated by $\bar{1}$, assigning a value for the generator is sufficient to define a homomomorphism. Namely, we have $\varphi(\bar{m}) = (\bar{m}, \bar{m})$ for $\bar{m} \in \mathbb{Z}_{10}$). Since $\varphi(\bar{m})\varphi(\bar{n}) = (\bar{m}, \bar{m})(\bar{n}, \bar{n}) = (\bar{m}\bar{n}, \bar{m}\bar{n}) = (\overline{mn}, \overline{mn}) = \varphi(\overline{mn})$, φ is a ring homomorphism. Suppose now $\varphi(\bar{m}) = \varphi(\bar{n})$, i.e., $(\bar{m}, \bar{m}) = (\bar{n}, \bar{n})$. Then $\bar{m} = \bar{n}$ in both $\mathbb{Z}_2$ and $\mathbb{Z}_5$. So, $2 \mid (m-n)$ and $5 \mid (m-n)$. Since 2 and 5 are relatively prime, $10 = 2 \cdot 5 \mid (m-n)$. Thus $\bar{m} = \bar{n}$ in $\mathbb{Z}_{10}$ and φ is injective. Since $|\mathbb{Z}_{10}| = |\mathbb{Z}_2 \times \mathbb{Z}_5| = 10$, φ is also surjective, hence an isomorphism. (cf. the last paragraph on page 62 and $\boxed{2.8.3}$ (p75)).

(b) Note, for any $\bar{m} \in \mathbb{Z}_2$ and $\bar{n} \in \mathbb{Z}_4$, we have $4(\bar{m}, \bar{n}) = (4\bar{m}, 4\bar{n}) = (\bar{0}, \bar{0})$ in $\mathbb{Z}_2 \times \mathbb{Z}_4$. Thus each element in $\mathbb{Z}_2 \times \mathbb{Z}_4$ is of (additive) order ≤ 4. In contrast, $\bar{1}$ and $\bar{n}$ in $\mathbb{Z}_8$ are of order 8. Therefore, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ are not isomorphic.