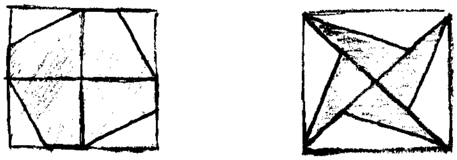
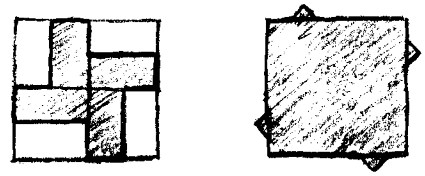
Homework 1

$\boxed{\#1}$ Give an example of a polygon with the symmetry group $C(4)$.



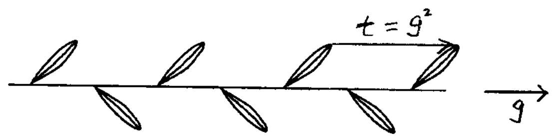
Note These are obtained, first by taking a square,



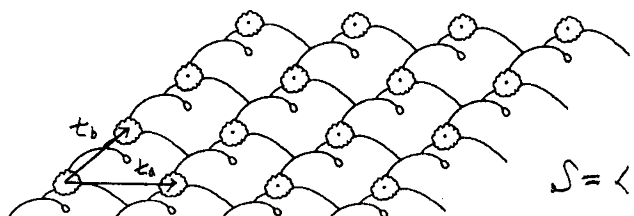
and then removing or adding 4 congruent pieces, so that it has no reflection symmetry.

$\boxed{5.1.3}$

ρ : rotations, t : translations, r : reflections, g : glide reflections



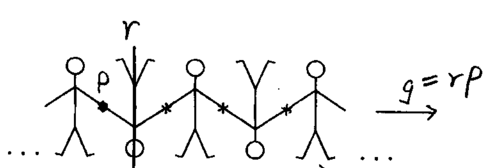
(1.4) **Figure.** $S = \langle g \rangle = \{\cdots g^{-1}, id, g, g^2, \cdots\} = \{\text{glides, translations}\}$



(1.5) **Figure.**

$$S = \langle t_a, t_b \mid t_a t_b = t_b t_a \rangle = \{ t_{na+mb} \mid n, m \in \mathbb{Z} \} = \{\text{translations}\}$$

(1.6) **Figure.** $S = \langle \rho_1, \rho_2 \mid \rho_1^2 = \rho_2^2 = 1 \rangle = \left\{ \begin{array}{l} \overbrace{\rho_1, \rho_2, \rho_1\rho_2\rho_1, \rho_2\rho_1\rho_2, \cdots}^{\text{rotations}} \\ \underbrace{\rho_1\rho_2, \rho_2\rho_1, \rho_1\rho_2\rho_1\rho_2, \cdots}_{\text{translations}} \end{array} \right\}$



(1.7) **Figure.** $t = g^2$

$$S = \langle r, \rho \mid r^2 = \rho^2 = 1 \rangle = \{ \rho, r\rho, \rho r \rho, r\rho r\rho, \cdots, r, \rho r, r\rho r, \rho r \rho r, \cdots \}$$

$= \{\text{reflections, rotations, translations, glides}\}$

Exercise : Identify which one is which!

$\boxed{5.1.4}$

See the first half of the proof of Theorem 3.4 (p165).

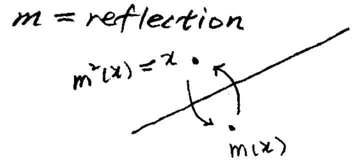
5.2.3

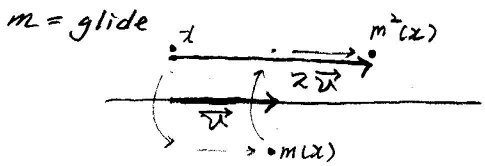
Recall the definition of a normal subgroup (Definition 4.8, p52) and its equivalent, $\boxed{2.4.13\ (b)}$ (p72). By Proposition 2.11 (p161), we have $t_p O t_p^{-1} = O' \neq O$ for $t_p \in M$, so O is not a normal subgroup of M.

5.2.4

Being orientation-reversing, m is either a reflection or a glide, and can be expressed as $m = t_a \rho_\theta r$ by (2.4) or as $m(z) = e^{i\theta}\bar{z} + \beta$ by $\boxed{5.2.15\ (b)}$.

i) If m is a reflection, then, obviously, m^2 is the identity. If m is a glide along $\vec{u}$, then m^2 is a translation by $2\vec{u}$.



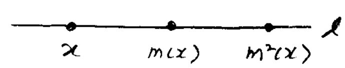


ii) $m^2 = (t_a \rho_\theta r)(t_a \rho_\theta r) = t_a \rho_\theta (t_{r(a)} r) \rho_\theta r = t_a (t_{\rho_\theta r(a)} \rho_\theta)(\rho_{-\theta} r) r = t_{a+\rho_\theta r(a)} \rho_{\theta-\theta} r^2 = t_{a+\rho_\theta r(a)}$ is a translation. (We used (2.5) (p159)).

iii) $m^2(z) = e^{i\theta}(\overline{e^{i\theta}\bar{z} + \beta}) + \beta = e^{i\theta}(e^{-i\theta} z + \bar{\beta}) + \beta = z + (e^{i\theta}\bar{\beta} + \beta)$ is a translation.

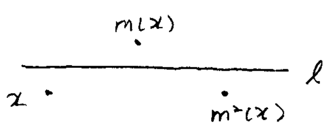
5.2.13

(a) ($\Longrightarrow$) Let $x \in \ell$.



Then $x, m(x), m^2(x)$ are colinear.

($\Longleftarrow$) If $x \notin \ell$, then



$x, m(x), m^2(x)$ are not colinear.

(b) Since $m^2(x) \neq x$, $m^2 \neq id$ and m is a glide reflection along a line, say, ℓ'. If $x \notin \ell'$, then $x, m(x), m^2(x)$ are not colinear by (a) ($\Longleftarrow$); a contradiction. So $x \in \ell'$. Similarly, noting m^{-1} is also a glide along ℓ', we get $m^2(x) \in \ell'$. Sharing two points x and $m^2(x)$ in common, the lines ℓ and ℓ' coincide. Thus m is a glide along ℓ.

5.3.1

Note, in D_n, $yx = x^{-1}y \Rightarrow yxy^{-1} = x^{-1} \Rightarrow yx^ky^{-1} = x^{-k} \Rightarrow yx^k = x^{-k}y$ for $k \in \mathbb{Z}$.

So we have

$$x^2 y x^{-1} y^{-1} x^3 y^3 \overset{y^{-1}=y}{=} x^2 y x^{-1} y x^3 y^3 \overset{yx^3 = x^{-3}y}{=} x^2 y x^{-1}(x^{-3}y) y^3 = x^2 y x^{-4} y^4 \overset{yx^{-4}=x^4y}{=} x^2 (x^4 y) y^4$$
$$= x^6 y^5 = x^6 y.$$

$\boxed{5.3.4}$

(a)

$1 \cdot \{1, x^5\} = \{1, x^5\} = \{1, x^5\} \cdot 1$ $\qquad$ $y\{1, x^5\} = \{y, x^5 y\} = \{1, x^5\} y$

$x \cdot \{1, x^5\} = \{x, x^6\} = \{1, x^5\} \cdot x$ $\qquad$ $xy\{1, x^5\} = \{xy, x^6 y\} = \{1, x^5\} xy$

$x^2 \cdot \{1, x^5\} = \{x^2, x^7\} = \{1, x^5\} x^2$ $\qquad$ $x^2 y\{1, x^5\} = \{x^2 y, x^7 y\} = \{1, x^5\} x^2 y$

$x^3 \cdot \{1, x^5\} = \{x^3, x^8\} = \{1, x^5\} x^3$ $\qquad$ $x^3 y\{1, x^5\} = \{x^3 y, x^8 y\} = \{1, x^5\} x^3 y$

$x^4 \{1, x^5\} = \{x^4, x^9\} = \{1, x^5\} x^4$ $\qquad$ $x^4 y\{1, x^5\} = \{x^4 y, x^9 y\} = \{1, x^5\} x^4 y.$

(b) By (a), $H = \{1, x^5\}$ is a normal subgroup of D_{10} and $|D_{10}/H| = 20/2 = 10$. Let $X = xH$ and $Y = yH$ be elements of D_{10}/H. We have $X^5 = (xH)^5 = x^5 H = x^5\{1, x^5\} = \{x^5, 1\} = H$ and $Y^2 = (yH)^2 = y^2 H = H$. Moreover, $YX = (yH)(xH) = (yx)H = (x^{-1}y)H = (x^{-1}H)(yH) = X^{-1}Y$. So X and Y satisfy the corresponding relations in D_5. Since X and Y generate D_{10}/H, we conclude $D_{10}/H \cong D_5$ by Proposition 3.6 (p165).

(c) Let $D_{10} = \langle x, y \mid x^{10} = 1, y^2 = 1, yx = x^{-1}y \rangle$, $D_5 = \langle X, Y \mid X^5 = 1, Y^2 = 1, YX = X^{-1}Y \rangle$, and $H \cong C(2) = \langle a \mid a^2 = 1 \rangle$. Set $A = (X, a) \in D_5 \times C(2)$ and $B = (Y, 1) \in D_5 \times C(2)$. Verify the followings: 1) $A^{10} = (1, 1)$, 2) $B^2 = (1, 1)$, 3) $BA = A^{-1}B$ (Note $A^{-1} = A^9 = (X^4, a)$ and $YX = X^{-1}Y = X^4 Y$) 4) A and B generate $D_5 \times C(2)$. By Proposition 3.6, the assignments $x \leftrightarrow A$ and $y \leftrightarrow B$ give an isomorphism between D_{10} and $D_5 \times C(2) \cong D_5 \times H$.

$\boxed{5.4.9}$

The possible symmetries are reflection r in the horizontal center line and translations generated by a unit translation t. Note r and t commute. So the symmetry group $S = \langle r, t \mid r^2 = 1, rt = tr \rangle = \{ rt^n, t^n \mid n \in \mathbb{Z} \}$. On the other hand, setting $C_2 = \langle y \mid y^2 = 1 \rangle$ and $C_\infty = \langle x \mid \phi \rangle$, we get $C_2 \times C_\infty = \{ (y, x^n), (1, x^n) \mid n \in \mathbb{Z} \} = \langle (y, 1), (1, x) \mid (y, 1)^2 = (1, 1), (y, 1)(1, x) = (1, x)(y, 1) \rangle$
So the correspondences $r \leftrightarrow (y, 1)$ and $t \leftrightarrow (1, x)$ give rise to an isomorphism between S and $C_2 \times C_\infty$.