Homework 5

$\boxed{6.1.6}$

(i) $1+1+1+2+5$: This is not a valid class equation, since it implies that $|Z(G)| = 3 \nmid 10$, a contradiction to $Z(G) \leq G$. cf. $\boxed{6.1.3}$

(ii) $1+2+2+5$: This is the class equation for $D_5 = \langle \rho, r \mid \rho^5 = 1, r^2 = 1, r\rho = \rho^{-1}r \rangle = \{1\} \cup \{\rho, \rho^4\} \cup \{\rho^2, \rho^3\} \cup \{r, \rho r, \rho^2 r, \rho^3 r, \rho^4 r\}$. cf. HW4 $\boxed{\#3}$

(iii) $1+2+3+4$: This is not valid, since $3 \nmid 10$. cf. (1.8) (p198)

(iv) $1+1+2+2+2+2$: This implies $|Z(G)| = 2$ and, for $x \in G \setminus Z(G)$, $|C_x| = 2$, i.e., $|Z(x)| = 5$; a contradiction since $Z(G) \leq Z(x)$ and $2 \nmid 5$. Therefore, this is not a valid class equation.

$\boxed{6.2.8}$

(a) Since $T \cong A_4$, one may refer to Practice Midterm #4. Here, we follow the arguments in our textbook (Section 6.2, pp 200~1). The rotational symmetry group T of a tetrahedron consists of the identity, three rotations by π (of order 2), four rotations by $2\pi/3$ (of order 3), and four rotations by $-2\pi/3$ (of order 3). So it has three subgroups of order 2, which are all conjugate being stabilizers of a transitive action (cf. Proposition 6.5 (b), p179). Thus, all three rotations by π form one conjugacy class. Verify all rotations by $2\pi/3$ are conjugate each other (Draw pictures). Since $8 \nmid 12$, all eight rotations by $\pm 2\pi/3$ cannot form a conjugacy class. So rotations by $2\pi/3$ form one conjugacy class of order 4 and rotations by $-2\pi/3$ form another of order 4. Therefore, the class equation of T is $12 = 1 + 3 + 4 + 4$.

(b) From (a) and $\boxed{6.1.3}$, we see $|Z(T)| = 1$, so $Z(T) = \{id\}$.

(c) Let H be a subgroup of T of order 4. Then elements of H are of order 1, 2, or 4. There are three elements of order 2 and no element of order 4 in T. These three order-2 elements together with the identity form the unique subgroup of order 4, H, which is isomorphic to the Klein four group.

(d) A subgroup K of T with $|K| = 6$ must be normal, since $[T:K] = 2$. cf. $\boxed{2.6.10(a)}$ (p74). By Lemma 2.5 (b) (p202), this can't happen.

7.1.1

Let $A = (a_{ij})$ and $B = (b_{ij})$. If we denote the standard basis for $\mathbb{R}^n$ by $e_1, \cdots, e_n$, then we have, for all $1 \le i, j \le n$, that

$$a_{ij} = e_i^t A e_j = e_i^t B e_j = b_{ij},$$

so $A = B$.

7.1.4

We apply the Gram-Schmidt procedure to the standard basis e_1, e_2 for $\mathbb{R}^2$.

i) normalize e_1 : $\langle e_1, e_1 \rangle = e_1^t A e_1 = A_{11} = 2$. Let $u_1 = e_1/\sqrt{2} = (1/\sqrt{2}, 0)^t$.

ii) find a vector u orthogonal to u_1 : $u = e_2 - \langle e_2, u_1 \rangle u_1 = e_2 - (e_2^t A u_1) u_1$
$= e_2 - (1/\sqrt{2}) u_1 = (-1/2, 1)$

iii) normalize u : $\langle u, u \rangle = u^t A u = (-1/2, 1) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = 3/2$.

Let $u_2 = u/\sqrt{3/2} = (-\sqrt{6}/6, 2\sqrt{6}/6)$.

Obviously, u_1 and u_2 form a basis for $\mathbb{R}^2$. Verify they are orthonormal.

<u>Exercise</u> Find the matrix P such that $A = P^t P$.

7.1.5

(a) For any $A \in Mat(n \times n)$, we have $A = (1/2)(A + A^t) + (1/2)(A - A^t)$.
Observe $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric. Suppose $A = A_s + A_K = B_s + B_K$, where A_s, B_s are symmetric and A_K, B_K are skew-symmetric. Then $A_s - B_s = (A_s - B_s)^t = (B_K - A_K)^t = -B_K + A_K$
$= B_s - A_s$, so $A_s = B_s$ and $A_K = B_K$. Thus, the expression above is unique.

(b). Define $(,)$ and $[,]$ by $(a, b) = (1/2)\langle a, b \rangle + (1/2)\langle b, a \rangle$ and $[a, b] = (1/2)\langle a, b \rangle - (1/2)\langle b, a \rangle$. Verify $(,)$ is symmetric and $[,]$ is skew-symmetric. Note $\langle a, b \rangle = (a, b) + [a, b]$.