# Homework 6

## 7.2.3

Recall Theorem 1.22. As in $\boxed{7.1.4}$, we apply the Gram-Schmidt orthogonalization to the standard basis for $\mathbb{R}^n$.

(a). $u_1 = e_1 = (1, 0)^t$.

$$u_2 = e_2 - \frac{\langle e_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = e_2 - (1/1)\, u_1 = (-1, 1)^t.$$

So $\{(1,0)^t, (-1,1)^t\}$ is an orthogonal basis.

(b) $u_1 = e_1 = (1, 0, 0)^t$.

$$u_2 = e_2 - \frac{\langle e_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = e_2 - 0 \cdot u_1 = e_2 = (0, 1, 0)^t$$

$$u_3 = e_3 - \frac{\langle e_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle e_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = e_3 - (1/1)\, u_1 - (1/2)\, u_2 = (-1, -\tfrac{1}{2}, 1)^t$$

So, $\{(1,0,0)^t, (0,1,0)^t, (-1, -\tfrac{1}{2}, 1)^t\}$ is an orthogonal basis.

## 7.2.9

The standard basis for $P$ is $\{1, x, x^2, \cdots, x^n\}$. Let $e_i = x^{i-1}$.

(a) $u_1 = e_1 = 1$, $\langle u_1, u_1 \rangle = \int_{-1}^{1} 1 \cdot 1\, dx = 2$. $w_1 = u_1/\sqrt{2} = 1/\sqrt{2}$.

$$u_2 = e_2 - \frac{\langle e_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = x - \frac{1}{2}\int_{-1}^{1} x \cdot 1\, dx = x.$$

$\langle u_2, u_2 \rangle = \int_{-1}^{1} x \cdot x\, dx = 2/3$. $w_2 = u_2/\sqrt{2/3} = \frac{\sqrt{3}}{\sqrt{2}} x$.

$\{1/\sqrt{2}, \sqrt{3}x/\sqrt{2}\}$ is an orthonormal basis.

(b) $$u_3 = e_3 - \frac{\langle e_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle e_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = x^2 - \left(\frac{1}{2}\int_{-1}^{1} x^2 \cdot 1\, dx\right) \cdot 1 - \left(\frac{3}{2}\int_{-1}^{1} x^2 \cdot x\, dx\right) x$$
$$= x^2 - 1/3$$

$\langle u_3, u_3 \rangle = \int_{-1}^{1} (x^2 - 1/3)^2\, dx = 2/5 - 4/9 + 2/9 = 8/45$.

$w_3 = u_3/\sqrt{8/45} = \sqrt{45/8}\,(x^2 - 1/3)$

$\{1/\sqrt{2}, \sqrt{3}x/\sqrt{2}, \sqrt{45/8}\,(x^2 - 1/3)\}$ is an orthonormal basis.

## 7.2.10

i) $\langle a_1 A_1 + a_2 A_2, B \rangle = tr((a_1 A_1 + a_2 A_2)^t B) = tr((a_1 A_1^t + a_2 A_2^t) B) =$
$tr(a_1 A_1^t B + a_2 A_2^t B) = a_1\, tr(A_1^t B) + a_2\, tr(A_2^t B) = a_1 \langle A_1, B \rangle + a_2 \langle A_2, B \rangle$.

$\langle A, b_1B_1 + b_2B_2 \rangle = tr(A^t(b_1B_1 + b_2B_2)) = tr(b_1A^tB_1 + b_2A^tB_2) =$
$b_1 tr(A^tB_1) + b_2 tr(A^tB_2) = b_1\langle A, B_1\rangle + b_2\langle A, B_2\rangle.$

Therefore, $\langle\ ,\ \rangle$ is a bilinear form on $V$. Let $A = (a_{ij}) \neq 0$. Then

$$\langle A, A\rangle = tr(A^tA) = \sum_{k=1}^{n}(A^tA)_{kk} = \sum_{k=1}^{n}\left(\sum_{l=1}^{n}(A^t)_{kl}A_{lk}\right) = \sum_{k=1}^{n}\sum_{l=1}^{n}(A_{lk})^2 > 0$$

Thus, $\langle\ ,\ \rangle$ is positive definite.

ii) The standard basis for $V$ is $\{e_{ij} \mid 1 \leq i, j \leq n\}$, where $e_{ij}$'s are matrix units ((2.5), p10). This turns out to be orthonormal, since

$$\langle e_{ij}, e_{kl}\rangle = tr(e_{ij}{}^t e_{kl}) = tr(e_{ji}e_{kl}) = tr(\delta_{ik}e_{jl}) \quad \boxed{1.2.8(a)}\ (p33)$$

$$= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise.}\end{cases}$$

Remark. Denote $A = (A_1, A_2, \cdots, A_n) = (a_{ij})$. We may identify $V = Mat(n\times n, \mathbb{R})$ with $\mathbb{R}^{n^2}$ by $A \longleftrightarrow (A_1{}^t, A_2{}^t, \cdots, A_n{}^t) = (a_{11}, a_{21}, \cdots, a_{n1}, a_{12}, \cdots, a_{n2}, \cdots, a_{nn}) \in \mathbb{R}^{n^2}$. Under this identification, the given bilinear form is nothing but the standard dot product on $\mathbb{R}^{n^2}$. In view of this fact, the above results are obvious.

$\boxed{7.2.11}$

$A$ is negative definite $\iff -A$ is positive definite

$\iff \det(-A)_i = (-1)^i \det A_i > 0$ for all $i$

$\iff sgn(\det A_i) = (-1)^i$ for all $i$.

$\boxed{7.2.15}$

(c) $v \in W_2^\perp \iff \langle v, w\rangle = 0,\ \forall w \in W_2 \implies \langle v, w\rangle = 0,\ \forall w \in W_1 \iff v \in W_1^\perp$. Thus, $W_2^\perp \subseteq W_1^\perp$.

(b). $v \in W \implies \langle v, w\rangle = 0.\ \forall w \in W^\perp \iff v \in (W^\perp)^\perp$, so $W \subseteq W^{\perp\perp}$

(a). $(\subseteq)$ Note $W_i \subseteq W_1 + W_2$. By (c), $(W_1 + W_2)^\perp \subseteq W_i^\perp$. Thus, we have $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$

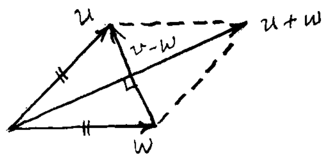
$(\supseteq)$ $v \in W_1^\perp \cap W_2^\perp \implies \langle v, w_1\rangle = 0,\ \forall w_1 \in W_1$ and $\langle v, w_2\rangle = 0,\ \forall w_2 \in W_2$

$\implies \langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle = 0.\ \forall w_1 \in W_1,\ \forall w_2 \in W_2$

$\implies v \in (W_1 + W_2)^\perp$

$\boxed{7.3.3}$

i) If $|u| = |w|$, then $\langle u+w, u-w \rangle = \langle u, u \rangle - \langle u, w \rangle + \langle w, u \rangle - \langle w, w \rangle$
$= |u|^2 - |w|^2 = 0$ since $\langle\ ,\ \rangle$ is symmetric. So $(u+w) \perp (u-w)$.

ii) 


Two diagonals of a rhombus are orthogonal.

$\boxed{7.3.11}$

Let $A = (a_{ij})$. Suppose, on the contrary, that $a_{k\ell}$ $(k > \ell)$ is a maximal matrix entry. Let $E_{1k}$ and $E_{2\ell}$ be elementary matrices of the form (2.6ii) (p11). By the lemma below, $B = E_{2\ell}{}^t E_{1k}{}^t A E_{1k} E_{2\ell}$ is positive definite. By Theorem 1.25, $\det B_2 > 0$. Observe, however, that $B_2 = \begin{pmatrix} a_{kk} & a_{k\ell} \\ a_{\ell k} & a_{\ell\ell} \end{pmatrix}$.

$$A = \begin{bmatrix} a_{\ell\ell} & a_{\ell k} \\ a_{k\ell} & a_{kk} \end{bmatrix}, \quad E_{1k}{}^t A E_{1k} = \begin{bmatrix} a_{kk} & a_{k\ell} & \\ a_{\ell k} & a_{\ell\ell} & \\ & & \end{bmatrix}, \quad B = \left[\begin{array}{cc|c} a_{kk} & a_{k\ell} & \\ a_{\ell k} & a_{\ell\ell} & \\ \hline & & \end{array}\right]$$

Since $a_{k\ell} = a_{\ell k} \geq a_{kk}, a_{\ell\ell}$, we have $\det B_2 = a_{kk} a_{\ell\ell} - a_{k\ell}^2 \leq 0$ ; a contradiction!

<u>Lemma</u> Let $P$ be invertible. Then $A$ is positive definite if and only if $P^t A P$ is positive definite.

Pf). $A$ is positive definite $\iff x^t A x > 0,\ \forall x \neq 0 \iff (Px)^t A (Px) = x^t (P^t A P) x > 0,\ \forall x \neq 0 \iff P^t A P$ is positive definite.

$\boxed{7.4.8}$

If $\langle\ ,\ \rangle$ were a hermitian form, its matrix (with respect to the standard basis for $\mathbb{C}^2$) would be hermitian, too. But, the matrix $\begin{pmatrix} 1 & i \\ -i & i \end{pmatrix}$ is not hermitian since one of its diagonal entries is not real.

$\boxed{7.4.10}$

Note i) $\det X^t = \det X$ (Proposition 3.18, p24) ii) $\det \overline{X} = \overline{\det X}$ (Prove)

Suppose $A$ is hermitian, i.e., $A^* = A$. Then we have

$\det A = \det A^* = \det \overline{A}^t = \det \overline{A} = \overline{\det A}$.

Therefore, $\det A$ must be a real number.