Homework 7

84

$\boxed{7.4.5}$

Let $W = \{ A \in Mat(n\times n, \mathbb{C}) \mid A = A^* \}$. To show W is a real vector space, one may verify all the axioms in Definition 1.6 (p80). Alternatively, one may regard $\mathbb{C}^{n^2} \cong Mat(n\times n, \mathbb{C})$ as a real vector space (why?) and then try to show W is a subspace of $Mat(n\times n, \mathbb{C})$, i.e., W satisfies (2.12)(a, b, c)(p86) with $F = \mathbb{R}$.

(a) Let $A = (a_{ij})$ and $B = (b_{ij})$ be elements of W. Since $(A+B)_{ij} = a_{ij} + b_{ij}$
$= \overline{a_{ji}} + \overline{b_{ji}} = \overline{(a_{ji} + b_{ji})} = \overline{(A+B)_{ji}}$, $A+B$ is hermitian, i.e., $A + B \in W$.

(b). If $A = (a_{ij}) \in W$ and $c \in \mathbb{R}$, then $(cA)_{ij} = c a_{ij} = c\,\overline{a_{ji}} \underset{\underset{c\in\mathbb{R}}{\uparrow}}{=} \bar{c}\,\overline{a_{ji}} = \overline{c a_{ji}}$
$= \overline{(cA)_{ji}}$, so $cA \in W$.

(c). Clearly, the zero matrix O is hermitian. So $O \in W$.
If $A = (a_{k\ell}) \in W$, then $A = \sum_{1\le k,\ell\le n} a_{k\ell} e_{k\ell}$ (p11) $= \sum_{k=1}^{n} a_{kk} e_{kk} + \sum_{1\le k<\ell\le n} (a_{k\ell} e_{k\ell} + a_{\ell k} e_{\ell k})$
$= \sum_k a_{kk} e_{kk} + \sum_{k<\ell} (a_{k\ell} e_{k\ell} + \overline{a_{k\ell}}\, e_{\ell k}) = \sum_k a_{kk} e_{kk} + \sum_{k<\ell} \{ (Re\, a_{k\ell} + i\, Im\, a_{k\ell}) e_{k\ell} +$
$(Re\, a_{k\ell} - i\, Im\, a_{k\ell}) e_{\ell k} \} = \sum_k a_{kk} e_{kk} + \sum_{k<\ell} (Re\, a_{k\ell})(e_{k\ell} + e_{\ell k}) + \sum_{k<\ell} (Im\, a_{k\ell})(i e_{k\ell} - i e_{\ell k})$.
Since $a_{kk}, Re\, a_{k\ell}, Im\, a_{k\ell} \in \mathbb{R}$, it is now clear $\{e_{kk}\}_{k=1}^{n} \cup \{e_{k\ell} + e_{\ell k}\}_{1\le k<\ell\le n} \cup$
$\{ i(e_{k\ell} - e_{\ell k}) \}_{1\le k<\ell\le n}$ form a basis for W. (What is $\dim_{\mathbb{R}} W$?)

$\boxed{7.4.6}$

Let V be a complex vector space and $\langle\, ,\, \rangle$ a hermitian form on V. Since (u_1, u_2) is an orthonormal basis for V, $\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = 1$, $\langle u_1, u_2 \rangle = 0$, and there are $a, b \in \mathbb{C}$ such that $u_2' = a u_1 + b u_2$. Therefore, we have

$(u_1, a u_1 + b u_2)$ is orthonormal

$\iff \langle u_1, u_1 \rangle = 1, \quad \langle u_1, a u_1 + b u_2 \rangle = 0, \quad \langle a u_1 + b u_2, a u_1 + b u_2 \rangle = 1$

$\iff a \cancel{\langle u_1, u_1 \rangle}^{1} + b \cancel{\langle u_1, u_2 \rangle}^{0} = 0$

$\quad |a|^2 \cancel{\langle u_1, u_1 \rangle}^{1} + \bar{a} b \cancel{\langle u_1, u_2 \rangle}^{0} + \bar{b} a \cancel{\langle u_2, u_1 \rangle}^{0} + |b|^2 \cancel{\langle u_2, u_2 \rangle}^{1} = 1$

$\iff a = 0$ and $|b| = 1$ (i.e. $b = e^{i\theta}$, $\theta \in \mathbb{R}$).

Thus, all orthonormal bases (u_1', u_2') with $u_1' = u_1$ are of the form $(u_1, e^{i\theta} u_2)$ for $\theta \in \mathbb{R}$.

7.4.12

($\Longrightarrow$) Let $(v_1, \cdots, v_n)$ be an orthonormal basis for V with respect to $\langle\, , \rangle$. Then, for any nonzero vector $v = \sum_{i=1}^{n} c_i v_i$ ($c_i \in \mathbb{C}$, $c_j \neq 0$ for some j), we have $\langle v, v \rangle = \sum_{i=1}^{n} |c_i|^2 \geq |c_j|^2 > 0$, so $\langle\, , \rangle$ is positive definite.

($\Longleftarrow$) The proof is identical to that of Theorem 1.22, except we define

$$w_k = v_k - \overline{\langle v_k, w_1 \rangle}\, w_1 - \overline{\langle v_k, w_2 \rangle}\, w_2 - \cdots - \overline{\langle v_k, w_{k-1} \rangle}\, w_{k-1}.$$

7.4.19

Note $\langle x_i, x_j \rangle \overset{\langle\,,\,\rangle \text{ is standard.}}{=} x_i^* x_j \overset{P \text{ is unitary}}{=} x_i^* (P^*P) x_j = (Px_i)^* (Px_j) = \langle Px_i, Px_j \rangle = \langle \lambda_i x_i, \lambda_j x_j \rangle = \bar{\lambda}_i \lambda_j \langle x_i, x_j \rangle$ for $i, j \in \{1, 2\}$. If $i = j$, then we have $|x_i|^2 = |\lambda_i|^2 |x_i|^2$. Since x_i is an eigenvector and hence $|x_i| \neq 0$, we get $|\lambda_i| = 1$ for $i = 1, 2$. Now, setting $i = 1$ and $j = 2$, we have $\langle x_1, x_2 \rangle = \bar{\lambda}_1 \lambda_2 \langle x_1, x_2 \rangle = (\lambda_2/\lambda_1) \langle x_1, x_2 \rangle$ for $\bar{\lambda}_1 \lambda_1 = |\lambda_1|^2 = 1$. Since $\lambda_1 \neq \lambda_2$, $\lambda_2/\lambda_1 \neq 1$, so we see $\langle x_1, x_2 \rangle = 0$.

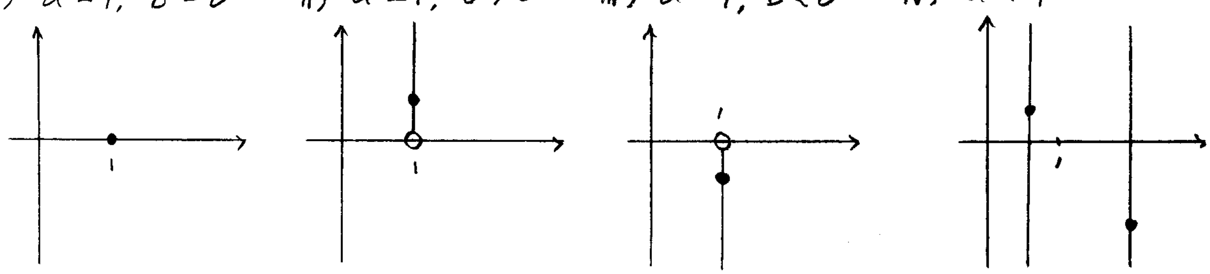
8.2.5

Let $X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ be elements of G. To determine the conjugacy class of A, we compute

$$XAX^{-1} = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/x & -y/x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xa & xb+y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/x & -y/x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & y(1-a)+xb \\ 0 & 1 \end{pmatrix}.$$

There are 4 types of conjugacy classes:

i) $a = 1,\ b = 0$ ii) $a = 1,\ b > 0$ iii) $a = 1,\ b < 0$ iv) $a \neq 1$



#1

We verify (5.1)(a)(b) (p176).

(a) For any $A \in \mathcal{S}$, $(I)(A) = IAI^t = A$

(b) For all $P, Q \in G$ and $A \in \mathcal{S}$, $(PQ)(A) = (PQ)A(PQ)^t = P(QAQ^t)P^t$
$= P\{(Q)(A)\}P^{-1} = (P)\{(Q)(A)\}$.

Therefore, the given map defines a group action.

$Stab(-I) = \{ P \in G \mid P(-I)P^t = -I \} = \{ P \in G \mid PP^t = I \} = O_n(F)$.

$\boxed{\#2}$

Let A be unitary, i.e., $AA^* = A^*A = I$.

i) $(-A)(-A)^* = AA^* = I$. $-A$ is unitary.

ii) $(\bar{A})(\bar{A})^* = \bar{A}\,\overline{A^*} = \overline{AA^*} = \bar{I} = I$. $\bar{A}$ is unitary.

iii) $(A^*)(A^*)^* = A^*A = I$. A^* is unitary

iv) $(A^3)(A^3)^* = (A^3)(A^*)^3 = A^2(AA^*)(A^*)^2 = A^2(A^*)^2 = A(AA^*)A^* =$
$AA^* = I$. A^3 is unitary.

v) $(I+A)(I+A)^* = (I+A)(I+A^*) = I + A + A^* + AA^* = 2I + A + A^*$
So, $I+A$ is not unitary unless $A + A^* = -I$