Homework 9

10.1.1

(b) $a+(-1)a = (1)a+(-1)a = (1+(-1))a = 0\cdot a = 0$. So $-a = (-1)a$.

(c) $(-a)b = ((-1)a)b = (-1)(ab) = -(ab)$

10.1.2

Let $R \subseteq \mathbb{C}$ be a subring and $\alpha = \sqrt[3]{2} \in R$. The argument on pp. 345-6 implies that $S := \{a\alpha^2 + b\alpha + c \mid a, b, c \in \mathbb{Z}\} \subseteq R$. Once we know the subset S is actually a subring, then S is the smallest subring containing α. It is clear that S is closed under addition and subtraction, and $1 \in S$. Since

$$(a_1\alpha^2 + b_1\alpha + c_1)(a_2\alpha^2 + b_2\alpha + c_2)$$
$$= (a_1a_2)\alpha^4 + (a_1b_2 + b_1a_2)\alpha^3 + (a_1c_2 + b_1b_2 + c_1a_2)\alpha^2 + (b_1c_2 + c_1b_2)\alpha + (c_1c_2)$$
$$= (a_1c_2 + b_1b_2 + c_1a_2)\alpha^2 + (2a_1a_2 + b_1c_2 + c_1b_2)\alpha + (2a_1b_2 + 2b_1a_2 + c_1c_2) \in S,$$

S is also closed under multiplication. Therefore, $S = \mathbb{Z}[\alpha]$ is the smallest subring of $\mathbb{C}$ containing α.

10.1.9

(a) ϕ is the additive identity. But if $A \neq \phi$, then $\phi \neq A \subseteq A \cup B$, so $A \cup B$ cannot be ϕ. Thus, the additive inverse $-A$ doesn't exist and R^+ is not a group. Therefore, R is not a ring.

(b). Addition is clearly commutative and ϕ is the additive identity. The additive inverse $-A$ is A. Check addition is associative. Thus, R^+ is an abelian group. Multiplication is clearly associative (and commutative) and U is the multiplicative identity. Check distributive laws. So R is a ring.

10.1.11

(a) $\bar{1}$ and $\overline{11} = -\bar{1}$ are units, since $\bar{1}\cdot\bar{1} = (-\bar{1})(-\bar{1}) = \bar{1}$. $\bar{5}$ and $\bar{7} = -\bar{5}$ are units, since $\bar{5}\cdot\bar{5} = (-\bar{5})(-\bar{5}) = \bar{1}$. None of the remaining elements of $\mathbb{Z}_{12}$ can be units (Verify). So the group G of units in $\mathbb{Z}_{12}$ is $\{\bar{1}, \bar{5}, \bar{7}, \overline{11}\}$ and, of course, $G \cong C_2 \times C_2$.

$\boxed{10.1.14}$

Since R^+ and $(R')^+$ are abelian groups, it is clear that $(R\times R')^+$ is also an abelian group (with the identity $(0,0)$). Associativity of multiplication and distributive laws follow from those of R and R'. $(1,1)$ is the multiplicative identity. Therefore, $R\times R'$ is a ring.

$\boxed{10.3.14}$

Let $\varphi : R \to R'$ be an isomorphism. Then φ induces a homomorphism $\overline{\varphi} : R[x] \to R'[x]$ defined by $\overline{\varphi}(\Sigma a_i x^i) = \Sigma \varphi(a_i) x^i$. (see the paragraph on page 354 after Proposition 3.4) It is now clear that $\overline{\varphi}$ is a bijection. Therefore, $R[x]$ and $R'[x]$ are isomorphic rings.

$\boxed{10.3.17}$

(a) $\varphi(1_R) = (1_R, 0) \neq (1_R, 1_{R'}) = 1_{R\times R'}$.

(b) $\varphi(r+s) = (r+s, r+s) = (r,r) + (s,s) = \varphi(r) + \varphi(s)$

$\varphi(rs) = (rs, rs) = (r,r)(s,s) = \varphi(r)\varphi(s)$

$\varphi(1_R) = (1_R, 1_R) = 1_{R\times R}$. $\quad \therefore \varphi$ is a homomorphism

(c) $\varphi((r_1,r_2)+(s_1,s_2)) = \varphi((r_1+s_1, r_2+s_2)) = r_1+s_1 = \varphi((r_1,r_2)) + \varphi((s_1,s_2))$.

$\varphi((r_1,r_2)(s_1,s_2)) = \varphi((r_1s_1, r_2s_2)) = r_1s_1 = \varphi((r_1,r_2))\varphi((s_1,s_2))$

$\varphi(1_{R\times R'}) = \varphi((1_R, 1_{R'})) = 1_R$ $\quad \therefore \varphi$ is a homomorphism

(d) $\varphi((r_1,r_2)+(s_1,s_2)) = (r_1+s_1)(r_2+s_2) \neq r_1r_2 + s_1s_2 = \varphi((r_1,r_2)) + \varphi((s_1,s_2))$.

(e) $\varphi((r_1,r_2)(s_1,s_2)) = r_1s_1 + r_2s_2 \neq (r_1+r_2)(s_1+s_2) = \varphi((r_1,r_2))\varphi((s_1,s_2))$.

$\boxed{\#1}$

(a) (Note $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.)

$(ij)^2 + (ji)^2 + (ik)^2 + (ki)^2 + (jk)^2 + (kj)^2 = k^2 + k^2 + j^2 + j^2 + i^2 + i^2 = -6$.

$kjkikjki = (-i)(j)(-i)(j) = k^2 = -1$

$k^6 j^3 i^2 + i^6 j^5 k^3 = (-1)^3(-j)(-1) + (-1)^3(j)(-k) = -j + jk = i - j$

(b) It is clear that conjugation is a bijection and $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$.

Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$. Then

$$q_1 q_2 = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\,i$$
$$+ (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\,j + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\,k$$

If we denote $q_1 = a_1 + u_1$ and $q_2 = a_2 + u_2$, then we've proved

$$q_1 q_2 = a_1a_2 - u_1 \circ u_2 + (a_1 u_2 + a_2 u_1 + u_1 \times u_2).$$

Therefore, we see

$$\begin{aligned}
\bar{q}_2 \bar{q}_1 &= (a_2 - u_2)(a_1 - u_1) \\
&= a_2 a_1 - u_2 \circ u_1 + (-a_2 u_1 - a_1 u_2 + (-u_2) \times (-u_1)) \\
&= a_1 a_2 - u_1 \circ u_2 - (a_1 u_1 + a_2 u_2 + u_1 \times u_2) \\
&= \overline{q_1 q_2}
\end{aligned}$$

(c). Let $q = a + bi + cj + dk = a + u$. Then,

$$\begin{aligned}
q\bar{q} &= (a + u)(a - u) \\
&= aa + u \circ u + (-au + au - \cancelto{0}{u \times u}) \\
&= a^2 + b^2 + c^2 + d^2.
\end{aligned}$$

Define $q\bar{q} = |q|^2$. Then $q(\bar{q}/|q|^2) = 1$ (if $q \neq 0$) and

$$q^{-1} = \bar{q}/|q|^2.$$