Homework 3

100

$\boxed{10.7.10}$

Suppose $M \subsetneq \overset{\text{ideal}}{I} \subseteq R$. Then, by assumption, I contains a unit and thus $I = R$. (cf. $\boxed{10.3.2}$). Therefore, M is maximal. Suppose now M' be a maximal ideal. Since $M' \neq R$, M' consists of non-units. So $R \setminus M' \supseteq \{\text{all units in } R\} = R \setminus M$, i.e., $M' \subseteq M$. Being maximal, $M' = M$.

$\boxed{10.7.11}$

$\bar{R} = R/P$ is an integral domain

$\iff P \subsetneq R$ and $[ab + P = (a+P)(b+P) = P \Rightarrow a + P = P$ or $b + P = P]$

$\iff P \subsetneq R$ and $[ab \in P \Rightarrow a \in P$ or $b \in P]$.

$\boxed{10.7.12}$

(a) (i) $P' \subsetneq R' \Rightarrow \varphi(1_R) = 1_{R'} \notin P' \Rightarrow \varphi^{-1}(P') \not\ni 1_R \Rightarrow \varphi^{-1}(P') \subsetneq R$.

(ii) $ab \in \varphi^{-1}(P') \Rightarrow \varphi(a)\varphi(b) = \varphi(ab) \in P' \Rightarrow \varphi(a) \in P'$ or $\varphi(b) \in P'$

$\Rightarrow a \in \varphi^{-1}(P')$ or $b \in \varphi^{-1}(P')$.

(b). Let $\varphi = \text{incl.} : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Since $\mathbb{Q}$ is a field, $(0) \subseteq \mathbb{Q}$ is the only (maximal) ideal of $\mathbb{Q}$. But $\varphi^{-1}((0)) = (0) \subseteq \mathbb{Z}$ is not maximal.

$\boxed{11.1.8}$

(a). Since $f(0) = f(1) = 1 \neq 0$, f has no linear factor. Since every reducible cubic polynomial contains a linear factor, f is irreducible.

(b). Note $f(1) = f(2) = 0$. So $f(x) = (x-1)(x-2)$.

(c). Since f has no roots in $\mathbb{F}_7$ and $\deg(f) = 2$, it is irreducible.

$\boxed{11.1.9}$

(a) Suppose $\{f_1, \cdots, f_k\}$ is the set of all monic irreducible polynomials in $F[x]$. Consider $f = f_1 \cdots f_k + 1$. Since $\deg(f) \geq k > 0$, f is not a constant. Since $f_i \nmid f$, $\forall i$, f is a monic irreducible polynomial by

Theorem (1.5)(c) (p391). Since $f \neq f_i,\ \forall i$, we got a contradiction.
(b) By $\boxed{10.2.6.(b)}$, $\{$units in $F[[x]]\} = \{f = \sum_{i=0}^{\infty} a_i x^i \mid a_0 \neq 0\}$ and $\{$nonunits in $F[[x]]\} = \{f = \sum_{i=0}^{\infty} a_i x^i \mid a_0 = 0\} \supset \{$irreducibles$\}$ $= (x)$. Thus, x is the only irreducible polynomial in $F[[x]]$, and since $x+1$ is a unit, the above argument fails.

$\boxed{11.1.13}$

(a). By division algorithm, we have $f/g = (qg+r)/g = q + r/g$ with $\deg r < \deg g$. Let $g(x) = c(x-a_1)^{n_1}\cdots(x-a_k)^{n_k}$. Then

$$\frac{r}{g} = \left(\frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1n_1}}{(x-a_1)^{n_1}}\right) + \cdots + \left(\frac{A_{k1}}{x-a_k} + \frac{A_{k2}}{(x-a_k)^2} + \cdots + \frac{A_{kn_k}}{(x-a_k)^{n_k}}\right)$$

for some $A_{ij} \in \mathbb{C}$.
(b). $\{x^i \mid i \geq 0\} \cup \left\{\frac{1}{(x-a)^n} \;\middle|\; n \geq 1,\ a \in \mathbb{C}\right\}$

$\boxed{11.2.8}$

Note $|11+7i|^2 = 160 < 325 = |18-i|^2$. We have

$18 - i = (1-i)(11+7i) + 3i$ and $|3i|^2 = 9$,
$11 + 7i = (2-4i)(3i) + (-1+i)$ and $|-1+i|^2 = 2$,
$3i = (1-i)(-1+i) + i$ and i is a unit.

cf. pp397~8.

Therefore, $\gcd(11+7i, 18-i) = 1$ (up to a unit).

$\boxed{11.2.10}$

Define $\sigma : F[[x]] \setminus \{0\} \longrightarrow \{0, 1, 2, \cdots\}$ by

$a_k \neq 0,\quad a_k x^k + a_{k+1} x^{k+1} + \cdots \longmapsto k.$

Verify that σ is a size function.

11.2.12

(a) Note $(a) \cap (b) = (m)$ for some $m \in R$ since R is a PID. Verify that $m = [a, b]$. If $(m) = (m')$, then m and m' are associates.

(b) Let $g = (a, b)$ then $a = a'g$ and $b = b'g$ for some a', b' with $(a', b') = 1$. Let $m = a''a = b''b$ for some a'', b'', $(a'', b'') = 1$. Then $a''a'g = b''b'g$, so $a''a' = b''b'$. Since $(a', b') = (a'', b'') = 1$, we see $a'' \sim b'$ and $ab = a'b'gg \sim a'a''gg = mg$.