Homework 6

12.2.6

Let R be a ring such that every finitely generated R-module is free, and assume R is not the zero ring. Suppose $\{0\} \neq I \subsetneq R$ is a proper ideal. Then there is $0 \neq a \in I$. Since R/I is an R-module which is finitely generated (i.e., by $1+I$), it is free by assumption. So there is an isomorphism $\varphi : R^n \longrightarrow R/I$ for some n. But we see $\varphi((a,0,\cdots,0)) = a \cdot \varphi((1,0,\cdots,0)) = \bar{0} \,(= 0 + I = I)$ since $a \in I$. This contradicts to the injectivity of φ and, thus, R has no proper ideal. By Proposition (3.16) (p357), R is a field.

12.4.1

(a) $\begin{pmatrix} 3 & 1 \\ \overline{1} & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 7 \\ \overline{1} & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 7 \\ \overline{1} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 4 & \overline{3} & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & \overline{3} & \overline{6} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{3} & \overline{6} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

12.4.6

(a) One can easily find the following identity:

$$QAP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \overline{8} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & \overline{3} & 3 & 1 \\ \overline{1} & 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & \overline{3} & \overline{3} & 10 \\ 0 & 0 & 6 & \overline{23} \\ 0 & 0 & 7 & \overline{27} \\ 0 & 1 & \overline{3} & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = A'$$

Since $Q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \overline{1} & 8 & 1 \end{pmatrix}$, the new basis for $\mathbb{Z}^3$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ \overline{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$,

and $\left\{ 1 \cdot \begin{pmatrix} 1 \\ 0 \\ \overline{1} \end{pmatrix}, 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}, 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is the basis for the submodule. Thus, this submodule is actually $\mathbb{Z}^3$ itself. (cf. p462).

$\boxed{12.\ 5.\ 1}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \mathbb{Z}/\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}.$ $\quad$ $\begin{pmatrix} 0 \\ 5 \end{pmatrix} \to \begin{pmatrix} 5 \\ 0 \end{pmatrix} \to \mathbb{Z}_5 \times \mathbb{Z}.$

$\begin{pmatrix} 2 & 0 & 0 \end{pmatrix} \to (2) \to \mathbb{Z}_2$ $\quad$ $\begin{pmatrix} 1 & 0 \\ \cancel{1} & \cancel{1} \end{pmatrix} \to (1) \to \mathbb{Z}/\mathbb{Z} \cong \{0\}.$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \mathbb{Z}$ $\quad$ $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & \bar{1} \\ \cancel{1} & \cancel{2} \end{pmatrix} \to (1) \to \{0\}$

$\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & \bar{4} \\ 1 & 4 \end{pmatrix} \to (4) \to \mathbb{Z}_4$ $\quad$ $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \cancel{2} & \cancel{1} \end{pmatrix} \to (0) \to \mathbb{Z}$

cf. Proposition (5.12) (p466).

$\boxed{12.\ 6.\ 2}$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to \begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \mathbb{Z}.$

$\boxed{12.\ 6.\ 4}$

Since $400 = 2^4 \cdot 5^2$, and $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$ (5 cases), $2 = 1+1$ (2 cases), there are $5 \times 2 = 10$ isomorphism classes.

$\mathbb{Z}_{16} \times \mathbb{Z}_{25}$	$\mathbb{Z}_{16} \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$	$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}.$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5.$

cf. Theorem (6.12) (p475).

$\boxed{12.\ 6.\ 5}$

Since $\mathbb{Z}_6$ is a PID (recall subgroups of a cyclic group are cyclic), we can apply Theorem (6.13) by the last paragraph of section 6 (p476). Observe proper ideals of $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \cdots, \bar{5}\}$ are $(\bar{2}) = (\bar{4}) \cong \mathbb{Z}_3$ and $(\bar{3}) \cong \mathbb{Z}_2$, so $\mathbb{Z}_6/(\bar{2}) \cong \mathbb{Z}_2$ and $\mathbb{Z}_6/(\bar{3}) \cong \mathbb{Z}_3$. Therefore, a finitely generated $\mathbb{Z}_6$-module is a direct sum of copies of $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$.

$\boxed{\#1}$

(i) $m, n \in Tor(M) \Rightarrow am = bn = 0$ for some $a, b \in R\setminus\{0\} \Rightarrow$ $ab(m+n) = 0$ and $ab \in R\setminus\{0\} \Rightarrow m+n \in Tor(M)$.

$m \in Tor(M)$, $r \in R \Rightarrow am = 0$ for some $a \in R\setminus\{0\} \Rightarrow a(rm) = r(am) = 0 \Rightarrow rm \in Tor(M)$.

Therefore, $Tor(M)$ is a submodule of M.

(ii) $Tor(R/I) = R/I$. $(0 \neq a \in I \subset R,\ a(r+I) = ar + I = I = \bar{0})$.

$Tor(I) = I \cap Tor(R)$

(iii) Define $\varphi : Tor(M \oplus N) \longrightarrow Tor(M) \oplus Tor(N)$ by $\varphi((m,n)) = (m,n)$. If $(m,n) \in Tor(M \oplus N)$, then $(0,0) = a(m,n) = (am, an)$ for some $a \in R\setminus\{0\}$. Thus, $m \in Tor(M)$, $n \in Tor(N)$ and so φ is really a map. Verify φ is an isomorphism.

(iv) Suppose $a(m + Tor(M)) = Tor(M)$ for some $a \in R\setminus\{0\}$. Then $am \in Tor(M)$, so $b(am) = 0$ for some $b \in R\setminus\{0\}$. Since $(ba)m = 0$ and $ba \in R\setminus\{0\}$, we see $m \in Tor(M)$. Thus, $Tor(M/Tor(M)) = \{Tor(M)\}$.

(v) The $\mathbb{Z}_2$-module $\mathbb{Z}_2$ has the required property.

$\boxed{\#2}$

(a), (b). Let $\{e_1, \cdots, e_n\}$ be the standard basis for R^n. Since f is surjective, there are $m_i \in M$ such that $f(m_i) = e_i$ $(1 \le i \le n)$. Define an R-module homomorphism $\varphi : R^n \longrightarrow M$ by $\varphi(e_i) = m_i$. Then, $f \circ \varphi = Id_{R^n}$. Define now $\Phi : \ker f \oplus R^n \longrightarrow M$ by $\Phi(a,b) = a + \varphi(b)$.

i) Φ is obviously an R-module homomorphism. ii) If $a + \varphi(b) = 0$, then $0 = f(a) = f(-\varphi(b)) = -b$, so $b = 0$ and $a = 0$. Thus, Φ is 1-1.

iii) For any $m \in M$, $\Phi(m - \varphi(f(m)), f(m)) = m$ and $m - \varphi(f(m)) \in \ker f$ since $f(m - \varphi(f(m))) = f(m) - f(m) = 0$. So Φ is onto. By i)~iii), Φ is an isomorphism.

(c). Let $R = \mathbb{Z}$ and $f = p : \underset{M}{\overset{\shortparallel}{\mathbb{Z}}} \longrightarrow \underset{N}{\overset{\shortparallel}{\mathbb{Z}_2}}$. Then $\ker p \oplus \mathbb{Z}_2 \cong 2\mathbb{Z} \oplus \mathbb{Z}_2 \cong \mathbb{Z} \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}$.