Homework 7

cf. Proposition (5.12) (p466) for $\boxed{12.6.1}$, $\boxed{12.6.3}$, $\boxed{12.6.7}$

12.6.1

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 .$$

12.6.3

(a) $$\begin{pmatrix} 3 & 2 \\ 2 & 0 \\ 8 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -4 \\ 0 & -4 \end{pmatrix} \to \begin{pmatrix} 4 \\ 4 \end{pmatrix} \to \begin{pmatrix} 4 \\ 0 \end{pmatrix} \to \mathbb{Z}_4 \times \mathbb{Z} .$$

(b) $$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \to \mathbb{Z}_2 \times \mathbb{Z}_2 .$$

(c) $$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 3 \\ 1 & -1 \\ 0 & 3 \end{pmatrix} \to \begin{pmatrix} 3 \\ 3 \end{pmatrix} \to \begin{pmatrix} 3 \\ 0 \end{pmatrix} \to \mathbb{Z}_3 \times \mathbb{Z} .$$

12.6.7

$$\begin{pmatrix} 1+i & 3 \\ 2-i & 5i \end{pmatrix} \to \begin{pmatrix} 1+i & 3 \\ 1+2i & -5 \end{pmatrix} \to \begin{pmatrix} 1+i & 3 \\ i & -8 \end{pmatrix} \to \begin{pmatrix} 1-i & 3 \\ 1 & -8 \end{pmatrix} \to \begin{pmatrix} 0 & 11-8i \\ 1 & -8 \end{pmatrix}$$

$\to (11-8i)$. $\quad V \cong \mathbb{Z}[i]/(11-8i)$.

12.m.8

Suppose $\mathbb{Q} = A \oplus B$ where A, B are proper subgroups of $\mathbb{Q}$. Then there are $0 \neq a/b \in A$ and $0 \neq c/d \in B$. Since A and B are $\mathbb{Z}$-modules, $(bc)(a/b) = ac \in A$ and $(da)(c/d) = ac \in B$, so $0 \neq ac \in A \cap B$; a contradiction to $A \cap B = \{0\}$. (cf. p472).

13.1.1

$a = 1/a \Rightarrow a^2 = 1 \Rightarrow 0 = a^2 - 1 = (a-1)(a+1). \Rightarrow a = \pm 1$ if $\operatorname{char} F \neq 2$, and $a = 1$ if $\operatorname{char} F = 2$.

13.1.3

Let R be an integral domain containing a field F as subring, then R is an F-module and, by assumption, R is free, i.e., $R \cong F^n$. Let $0 \neq a \in R$ and define $m_a : R \to R$ by $m_a(r) = ar$. Clearly, m_a is an F-module homomorphism (or, in other words, a linear operator on a vector space). If $m_a(r) = 0$, then $ar = 0$. Since $a \neq 0$ and R is an integral domain, $r = 0$ and m_a is injective. By the dimension formula (Theorem (1.6), p110), m_a is surjective, too. Thus, there is $r \in R$ such that $1 = m_a(r) = ar$, i.e., $r = a^{-1}$. Therefore, R is a field.

13.1.4

Recall that char F is the smallest positive integer n such that $n \cdot 1_F = 0$ (p358). Since F is an abelian group of order 8, we see $8 \cdot 1_F = 0$, so $n \mid 8$. Since n is prime (if $n = rs$ with $0 < r, s < n$, then $n \cdot 1_F = (r \cdot 1_F)(s \cdot 1_F) = 0$), $n = 2$.