Homework 8

13.2.1

Let $\alpha = \sqrt[3]{2}$ and $\beta = 1 + \alpha^2$, then $4 = \alpha^6 = (\beta - 1)^3 = \beta^3 - 3\beta^2 + 3\beta - 1$. So let $f(x) = x^3 - 3x^2 + 3x - 5$ and then $f(\beta) = 0$. Note f is irreducible over $\mathbb{Z}$ because it has no root in $\mathbb{Z}$. By Prop (3.6) (p400), the primitive polynomial f is irreducible over $\mathbb{Q}$.

13.2.3

Let $\alpha = \sqrt{3} + \sqrt{5}$, $\beta = \sqrt{3} - \sqrt{5}$, and $f(x) = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$.
Then $f_1(x) = (x-\alpha)(x+\alpha) = x^2 - (8 + 2\sqrt{15})$, $f_2(x) = (x-\alpha)(x-\beta) = x^2 - 2\sqrt{3}x - 2$, and $f_3(x) = (x-\alpha)(x+\beta) = x^2 - 2\sqrt{5}x + 2$.

(a) $irr(\alpha, \mathbb{Q}) = f$: f has no root in $\mathbb{Q}$ and $f_i \notin \mathbb{Q}[x]$.

(b) $irr(\alpha, \mathbb{Q}(\sqrt{5})) = f_3$: f_3 has no root in $\mathbb{Q}(\sqrt{5})$.

(c) $irr(\alpha, \mathbb{Q}(\sqrt{10})) = f$: f has no root in $\mathbb{Q}(\sqrt{10})$ and $f_i \notin \mathbb{Q}(\sqrt{10})[x]$.

(d) $irr(\alpha, \mathbb{Q}(\sqrt{15})) = f_1$: f_1 has no root in $\mathbb{Q}(\sqrt{15})$.

13.2.5

$$0 = f(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$$
$$\Longrightarrow \alpha(\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \cdots + a_1) = -a_0$$
$$\Longrightarrow \alpha^{-1} = -\frac{1}{a_0}(\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \cdots + a_1)$$

Note that $a_0 \neq 0$ since f is irreducible over F.

13.3.2

Note that ζ is a root of $x^7 - 1 = (x-1)(x^6 + x^5 + \cdots + x + 1)$ and η is a root of $x^5 - 1 = (x-1)(x^4 + \cdots + x + 1)$. By Corollary (4.6) (p405), $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 6$ and $[\mathbb{Q}(\eta) : \mathbb{Q}] = 4$. Therefore, if we had $\eta \in \mathbb{Q}(\zeta)$, then $6 = [\mathbb{Q}(\zeta) : \mathbb{Q}] = [\mathbb{Q}(\zeta) : \mathbb{Q}(\eta)][\mathbb{Q}(\eta) : \mathbb{Q}] = 4 \cdot [\mathbb{Q}(\zeta) : \mathbb{Q}(\eta)]$, so we would get a contradiction that $4 \mid 6$.

$\boxed{13.3.3}$

(a) x^2+1 (b) x^2-x+1 (c) x^4+1 (d) x^6+x^3+1.

(e) $x^4-x^3+x^2-x+1$ (f) x^4-x^2+1.

$\boxed{13.3.7}$

(a) Suppose $i \in \mathbb{Q}(\sqrt{-2})$, then $\mathbb{Q}(\sqrt{-2}) \ni i\sqrt{-2} = i(i\sqrt{2}) = -\sqrt{2}$, so we have $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{-2})$. Since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 = [\mathbb{Q}(\sqrt{-2}) : \mathbb{Q}]$, $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{-2})$, which is a contradiction to Proposition (2.9) (p495) because $x^2-2 \neq x^2+2$.

(b) Let $K = \mathbb{Q}(\sqrt[4]{-2})$ and $F = \mathbb{Q}(\sqrt{-2})$. Then $[K : F] = 2$ since $\mathrm{irr}(\sqrt[4]{-2}, F) = x^2 - \sqrt{-2}$. Suppose $i \in K$. Since $\sqrt{-2} \in F < K$, we have $K \ni i\sqrt{-2} = -\sqrt{2}$. But $\sqrt{2} \notin F$ by (a). Thus, we have $F \subsetneq F(\sqrt{2}) \subseteq K$ and $[K : F] = 2 = [F(\sqrt{2}) : F]$. Therefore, $F(\sqrt{2}) = K$; a contradiction because $\mathrm{irr}(\sqrt{2}, F) = x^2 - 2 \neq x^2 - \sqrt{-2}$.

(c) If $i \in \mathbb{Q}(\alpha)$, then $3 = [\mathbb{Q}(\alpha), \mathbb{Q}] = [\mathbb{Q}(\alpha), \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = 2 \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}(i)]$, so $2 \mid 3$; a contradiction.

$\boxed{13.4.7}$

If possible, then $\sqrt{\pi}$ would be a constructible number, hence algebraic over $\mathbb{Q}$ by Corollary (4.10) (p504). Then, by Theorem (3.10) (p499), $\pi = \sqrt{\pi} \cdot \sqrt{\pi}$ would be algebraic over $\mathbb{Q}$, a contradiction.

$\boxed{13.4.10}$

Let $p = 2^r + 1$ be a prime number. Then $2^r \equiv -1 \pmod{p}$, so we have $|2| = 2r$ in the multiplicative group $(\mathbb{Z}_p)^\times$. Thus $2r \mid p-1 = 2^r$ and $r = 2^k$ for some k.

$\boxed{13.4.11}$

Let $(\mathbb{Q} \subset) S \subset \mathbb{R}$ be a subfield of $\mathbb{R}$ with the said property and $a \in C$. By Theorem (4.9) (p504), there is a chain of subfields $\mathbb{Q} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n \ni a$, where $F_{i+1} = F_i(\sqrt{r_i})$, $F_i \ni r_i > 0$.

Since $r_0 \in \mathbb{Q} \subset \mathcal{S}$, $\sqrt{r_0} \in \mathcal{S}$. Thus, $F_1 = F_0(\sqrt{r_0}) = \mathbb{Q}(\sqrt{r_0}) \subset \mathcal{S}$. Since $r_1 \in F_1 \subset \mathcal{S}$, $\sqrt{r_1} \in \mathcal{S}$. Thus, $F_2 = F_1(\sqrt{r_1}) \subset \mathcal{S}$. Inductively, we finally get $a \in F_n \subset \mathcal{S}$. Therefore $\mathcal{C} \subset \mathcal{S}$.