

# 250A Homework 1

Solution by Jaejeong Lee

**Question 1** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear operator. Show that  $T$  has 1 and 2 dimensional invariant subspaces.

**Solution** The characteristic polynomial of  $T$  is real cubic, so it has at least one real root  $\lambda$  and an eigenvector  $v \in \mathbb{R}^3$  such that  $Tv = \lambda v$ . Note that  $\text{span}\{v\} \subset \ker(T - \lambda I)$  is a 1-dimensional invariant subspace of  $\mathbb{R}^3$  and  $\text{rank}(T - \lambda I) = \dim \text{im}(T - \lambda I) = 3 - \dim \ker(T - \lambda I) < 3$ . Now if  $\text{rank}(T - \lambda I) = 0$ , then  $T = \lambda I$  and any 2-dimensional subspace of  $\mathbb{R}^3$  is invariant under  $T$ . If  $\text{rank}(T - \lambda I) = 1$ , then  $\ker(T - \lambda I)$  is a 2-dimensional invariant subspace, since for  $v \in \ker(T - \lambda I)$  we have  $(T - \lambda I)(Tv) = T(Tv - \lambda v) = T(0) = 0$  and hence  $T(\ker(T - \lambda I)) \subset \ker(T - \lambda I)$ . If  $\text{rank}(T - \lambda I) = 2$ , then  $\text{im}(T - \lambda I)$  is a 2-dimensional invariant subspace since  $T((T - \lambda I)v) = (T - \lambda I)(Tv) \in \text{im}(T - \lambda I)$ . Therefore, in any case,  $T$  has a 2-dimensional invariant subspace, too.

**Question 2** Let  $M$  be the space of 3 matrices. What is  $\dim(M)$ ? Now define the linear operator  $T : M \rightarrow M$  by

$$M \ni m \xrightarrow{T} \frac{1}{2} \left[ \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} m + m \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} \right].$$

Compute  $\det T$ .

**Solution** The standard basis for  $M$  is  $\{e_{ij} \mid 1 \leq i, j \leq 3\}$ , where  $e_{ij}$  is a matrix unit with its only nonzero entry being 1 at  $(i, j)$ . Thus  $\dim(M) = 9$ . We compute

$$T \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2m_{11} & 3m_{12} & 2m_{13} \\ 3m_{21} & 4m_{22} & 3m_{23} \\ 2m_{31} & 3m_{32} & 2m_{33} \end{pmatrix},$$

so the eigenvectors of  $T$  are  $\{e_{ij} \mid 1 \leq i, j \leq 3\}$  and the corresponding eigenvalues are  $\{1, \frac{3}{2}, 1, \frac{3}{2}, 2, \frac{3}{2}, 1, \frac{3}{2}, 1\}$ . Therefore,  $\det T = \prod(\text{eigenvalues}) = \frac{81}{8}$ .

**Question 3 Van der Monde Determinant.** Let  $A$  be the  $(n \times n)$  matrix with entries

$$A_{ij} = (a_i)^{j-1}.$$

Show that  $\det A = \prod_{i < j} (a_j - a_i)$ .

**Solution** Note that

$$A = \begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & (a_n)^2 & \cdots & (a_n)^{n-1} \end{pmatrix}.$$

We use induction on  $n$ . When  $n = 2$ , we verify

$$\det \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1.$$

Assume now the assertion is true up to  $n - 1$  and let

$$f(x) = \det \begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & (a_{n-1})^2 & \cdots & (a_{n-1})^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{pmatrix}.$$

Since  $f(x)$  is a polynomial of degree  $n - 1$  and  $f(a_i) = 0$  for  $1 \leq i \leq n - 1$ , we have

$$\begin{aligned} & f(x) \\ &= (-1)^{n+n} \det \begin{pmatrix} 1 & a_1 & \cdots & (a_1)^{n-2} \\ 1 & a_2 & \cdots & (a_2)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & \cdots & (a_{n-1})^{n-2} \end{pmatrix} (x - a_1)(x - a_2) \cdots (x - a_{n-1}) \\ &= \left( \prod_{1 \leq i < j \leq n-1} (a_j - a_i) \right) (x - a_1)(x - a_2) \cdots (x - a_{n-1}), \end{aligned}$$

by the induction hypothesis. Therefore, we finally get

$$\begin{aligned}
 & \det \begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & (a_n)^2 & \cdots & (a_n)^{n-1} \end{pmatrix} \\
 &= f(a_n) \\
 &= \left( \prod_{1 \leq i < j \leq n-1} (a_j - a_i) \right) (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \\
 &= \prod_{1 \leq i < j \leq n} (a_j - a_i)
 \end{aligned}$$

and the assertion is true for  $n$ .

**Question 4** (*Anti*)commutators. Let  $V$  be a finite dimensional vector space. Show that the mapping

$$[\cdot, \cdot] : L(V) \times L(V) \rightarrow L(V)$$

where

$$[\cdot, \cdot] : (M, N) \mapsto MN - NM \equiv [M, N],$$

obeys the Leibnitz rule  $[M, NR] = [M, N]R + N[M, R]$ . In addition, verify the Jacobi identity

$$[M, [N, R]] + [N, [R, M]] + [R, [M, N]] = 0.$$

Generalize the above laws to the mapping  $\{\cdot, \cdot\} : (M, N) \mapsto MN + NM \equiv \{M, N\}$ . Include also new rules which mix both operations.

**Solution** We want to find a kind of the Leibnitz rule

$$F(N * R) = F(N) * R + N * F(R)$$

in cases  $F(\cdot) = [M, \cdot]$  or  $\{M, \cdot\}$  and  $N * R = NR$ ,  $[N, R]$ , or  $\{N, R\}$ . For example, when  $F(\cdot) = [M, \cdot]$  and  $N * R = NR$ , we have

$$[M, NR] = [M, N]R + N[M, R] \tag{1}$$

and when  $F(\cdot) = [M, \cdot]$  and  $N * R = [N, R]$ , we have

$$[M, [N, R]] = [[M, N], R] + [N, [M, R]]. \quad (2)$$

Now I present a *trick* to find such rules. (It may not be a trick but a principle. Because I don't know why it works, to me, it is a trick.)

Matrices can be assigned two attributes, even and odd. If  $M$  is assigned odd, we mark it  $M'$ , and if even, leave it as it is.

**Step1** Write down (1) or (2).

e.g.  $[M, NR] = [M, N]R + N[M, R]$

**Step2** Assign attributes to  $M, N, R$  arbitrarily.

e.g.  $[M', N'R] = [M', N']R + N'[M', R]$

**Step3** Regard (odd,odd)-pair and (even,even)-pair as even and other pairs as odd.

e.g.  $N'R$  is odd, so  $[M', N'R]$  is even.  $[M', N']$  is even and  $[M', R]$  is odd.

**Step4** If you find [odd, odd], replace  $[\cdot, \cdot]$  by  $\{\cdot, \cdot\}$ .

e.g.  $\{M', N'R\} = \{M', N'\}R + N'[M', R]$

**Step5** If an odd, as an operator, goes past another odd, then replace  $+$  by  $-$ .

e.g.  $\{M', N'R\} = \{M', N'\}R - N'[M', R]$  ( $M'$  goes past  $N'$ )

**Step6** Remove markings  $'$  and declare you found a rule.

e.g.  $\{M, NR\} = \{M, N\}R - N\{M, R\}$

**Example1.**  $[M, NR] = \{M, N\}R - N\{M, R\}$ .

**Step1**  $[M, NR] = [M, N]R + N[M, R]$ .

**Step2**  $[M', N'R'] = [M', N']R' + N'[M', R']$

**Step3** Observe  $[M', N']$  and  $[M', R']$ .

**Step4**  $[M', N'R'] = \{M', N'\}R' + N'\{M', R'\}$

**Step5**  $[M', N'R'] = \{M', N'\}R' - N'\{M', R'\}$

**Step6**  $[M, NR] = \{M, N\}R - N\{M, R\}$

**Example2.**  $[M, \{N, R\}] = [\{M, N\}, R] - [N, \{M, R\}]$ .

**Step1**  $[M, [N, R]] = [[M, N], R] + [N, [M, R]]$ .

**Step2**  $[M', [N', R']] = [[M', N'], R'] + [N', [M', R']]$

**Step3** Observe  $[N', R']$ ,  $[M', N']$ , and  $[M', R']$ .

**Step4**  $[M', \{N', R'\}] = [\{M', N'\}, R'] + [N', \{M', R'\}]$

**Step5**  $[M', \{N', R'\}] = [\{M', N'\}, R'] - [N', \{M', R'\}]$

**Step6**  $[M, \{N, R\}] = [\{M, N\}, R] - [N, \{M, R\}]$

**Example3.**  $[M, \{N, R\}] = \{[M, N], R\} + \{N, [M, R]\}.$

**Step1**  $[M, [N, R]] = [[M, N], R] + [N, [M, R]].$

**Step2**  $[M, [N', R']] = [[M, N'], R'] + [N', [M, R']]$

**Step3**  $[N', R']$  is even,  $[M, N']$  and  $[M, R']$  are odd.

**Step4**  $[M, \{N', R'\}] = \{[M, N'], R'\} + \{N', [M, R']\}$

**Step5**  $[M, \{N', R'\}] = \{[M, N'], R'\} + \{N', [M, R']\}$  (No change)

**Step6**  $[M, \{N, R\}] = \{[M, N], R\} + \{N, [M, R]\}$

Find more on your own. Note that you never get  $\{M, \{N, R\}\}$  because of **Step3** and **Step4**.

**Question 5 Baker Campbell Hausdorff Formula.** Let  $V$  be a finite dimensional vector space and  $M, N \in L(V)$ . Show that

$$\exp(M) \exp(N) = \exp\left(M + N + \frac{1}{2}[M, N]\right),$$

if  $0 = [M, [M, N]] = [N, [M, N]]$ . *Hint: Develop and solve a differential equation for  $R(\lambda) \equiv \exp(\lambda M) \exp(\lambda N) \in L(V)$ .*

**Solution** We need the identity  $e^{\lambda M} N e^{-\lambda M} = e^{\lambda[M, \cdot]} N$  (proved below) to see

$$\begin{aligned} \frac{d}{d\lambda} R(\lambda) &= M R(\lambda) + e^{\lambda M} N e^{-\lambda M} R(\lambda) \\ &= (M + e^{\lambda[M, \cdot]} N) R(\lambda) \\ &= (M + N + \lambda[M, N]) R(\lambda), \end{aligned}$$

the last equality coming from the commutativity assumptions. Since  $R(0) = 0$  we get the unique solution

$$R(\lambda) = e^{\lambda(M+N) + \frac{1}{2}\lambda^2[M, N]}.$$

(Verify this. We need the commutativity assumptions again. To solve  $f'(x) = (a + bx)f(x)$  we observe  $(\log f(x))' = a + bx$ .) Plugging  $\lambda = 1$  we get the desired equality. To show  $e^{\lambda M} N e^{-\lambda M} = e^{\lambda[M, \cdot]} N$ , let  $L_M(\cdot) = [M, \cdot]$  and

compute the Taylor expansion for  $e^{\lambda M} N e^{-\lambda M}$ ;

$$\begin{aligned}\frac{d}{d\lambda} e^{\lambda M} N e^{-\lambda M} &= M e^{\lambda M} N e^{-\lambda M} - e^{\lambda M} N e^{-\lambda M} M \\ &= [M, e^{\lambda M} N e^{-\lambda M}] \\ &= L_M(e^{\lambda M} N e^{-\lambda M}),\end{aligned}$$

$$\begin{aligned}\frac{d^2}{d\lambda^2} e^{\lambda M} N e^{-\lambda M} &= [M, \frac{d}{d\lambda} e^{\lambda M} N e^{-\lambda M}] \\ &= L_M L_M(e^{\lambda M} N e^{-\lambda M}),\end{aligned}$$

and so on. Therefore

$$\begin{aligned}e^{\lambda M} N e^{-\lambda M} &= N + \lambda L_M(N) + \frac{1}{2} \lambda^2 L_M L_M(N) + \dots \\ &= e^{\lambda L_M} N.\end{aligned}$$