

250A Homework 1

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Question 1 What is the minimal number of generators in the direct sum of cyclic groups of order 3, 5 and 25?

Solution Denote $G = \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{25}$. First off, G cannot be generated by one element because the order of an element of G is at most $\text{lcd}(3, 5, 25) = 75$, which is less than $|G| = 375$. One can easily check, however, that G is generated by two elements $(1, 1, 0)$ and $(0, 0, 1)$. Therefore, the minimal number of generators in G is two.

Question 2 Prove that the group of rational numbers with respect to addition is not finitely generated.

Solution Suppose \mathbb{Q} is generated by a finite set $\{\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k} \mid p_i \in \mathbb{Z} \text{ and } q_i \in \mathbb{Z}^{>0}\}$. Denote $q = q_1 \cdots q_k$ and $\hat{q}_i = \frac{q}{q_i}$. Because $\frac{1}{q+1} \in \mathbb{Q}$, there are $m_i \in \mathbb{Z}$ such that $m_1(\frac{p_1}{q_1}) + \dots + m_k(\frac{p_k}{q_k}) = \frac{1}{q+1}$. This is absurd, however, since we then have

$$\frac{q}{q+1} = m_1 p_1 \hat{q}_1 + \dots + m_k p_k \hat{q}_k \in \mathbb{Z}.$$

Question 3 Describe all endomorphisms and all automorphisms of the group of rational numbers (with respect to addition).

Solution (a) Let $f \in \text{End}(\mathbb{Q}, +)$ and m be a positive integer. Because f is a group homomorphism, we have $f(m) = f(1 + \dots + 1) = f(1) + \dots + f(1) = mf(1)$ and $f(-m) = -f(m) = -(mf(1)) = m(-f(1)) = (-m)f(1)$. Now, any element of \mathbb{Q} is expressed as $\frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^{>0}$. Again, f being a group homomorphism, we have $f(m) = f(n\frac{m}{n}) = f(\frac{m}{n} + \dots + \frac{m}{n}) = f(\frac{m}{n}) + \dots + f(\frac{m}{n}) = nf(\frac{m}{n})$, thus

$$f\left(\frac{m}{n}\right) = \frac{f(m)}{n} = \frac{mf(1)}{n},$$

where the last equality comes from the previous discussion. Therefore, an endomorphism f of \mathbb{Q} is completely determined by $f(1)$, its value on 1, which can be an arbitrary element of \mathbb{Q} .

(b) Let $f \in \text{End}(\mathbb{Q}, +)$. Then f is invertible if and only if $f(1) \neq 0$. In this case the inverse f^{-1} is given by

$$f^{-1}\left(\frac{p}{q}\right) = \frac{p(f(1))^{-1}}{q}.$$

If $f, g \in \text{Aut}(\mathbb{Q}, +)$, then $(f \circ g)\left(\frac{m}{n}\right) = \frac{m(g(1)f(1))}{n}$. Therefore, we see that $\text{Aut}(\mathbb{Q}, +) \cong (\mathbb{Q}^*, \cdot)$.

Question 4 Describe all endomorphisms and all automorphisms of the direct sum of several copies of infinite cyclic group.

Solution Let $G = \sum_{i \in I} \mathbb{Z}_i$, where I is some index set and $\mathbb{Z}_i = \mathbb{Z}$. For each $k \in I$, let α_k be the element $\{u_i\}_{i \in I}$ of G such that $u_i = 0$ for $i \neq k$ and $u_k = 1$. Then $A = \{\alpha_i \mid i \in I\}$ is a basis of G ; we have that (1) $G = \langle A \rangle$ and (2) for distinct $\alpha_1, \dots, \alpha_k \in A$ and $n_i \in \mathbb{Z}$, $n_1\alpha_1 + \dots + n_k\alpha_k = 0$ implies $n_i = 0$ for every i . Such an abelian group with a basis as G is called *free abelian*.

(a) It is clear that every nonzero element α of G may be written uniquely in the form $\alpha = n_1\alpha_1 + \dots + n_k\alpha_k$. Then, every $f \in \text{End}(G)$ is completely determined by its values on A . Therefore, as a set, $\text{End}(G) = \prod_{i \in I} G_i$, where $G_i = G$, and obviously, $f \in \text{Aut}(G)$ iff $f(A) = \{f(\alpha_i) \mid i \in I\}$ is a basis of G .

(b) We consider the case when $I = \{1, \dots, n\}$ is finite. Then $\text{End}(G) = \text{Mat}_{n \times n}(\mathbb{Z})$. Suppose now that $f \in \text{Aut}(G)$. Because A and $f(A)$ are bases of G , there are some integers p_{ij} and q_{ij} such that $f(\alpha_j) = p_{1j}\alpha_1 + \dots + p_{nj}\alpha_n$ and $\alpha_j = q_{1j}f(\alpha_1) + \dots + q_{nj}f(\alpha_n)$. From these equalities, by setting $P = (p_{ij})$ and $Q = (q_{ij})$, we see that $PQ = QP = I_n$, i.e. $P, Q \in \text{GL}(n, \mathbb{Z})$. Therefore, we conclude that $\text{Aut}(G) \cong \text{GL}(n, \mathbb{Z})$.

Question 5 Calculate the number of homomorphisms of the cyclic group of order 4 into the symmetric group S_5 .

Solution Let $C_4 = \{a \mid a^4 = 1\}$ be the cyclic group of order 4 generated by a . A homomorphism $f \in \text{Hom}(C_4, S_5)$ is determined by its value on a and, because $a^4 = 1$, f satisfies $f(a)^4 = f(a^4) = f(1) = 1$. Thus the order of $f(a)$ must be 1, 2, or 4. One can check that S_5 has 1 element of order 1, 25 elements of order 2, and 30 elements of order 4. Therefore, $|\text{Hom}(C_4, S_5)| = 1 + 25 + 30 = 56$.