250A Homework 3

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Exercise 3.4.2. Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

Solution See [DF, page 69] for the lattice diagrams of subgroups of Q_8 and D_8. Each possible path from top to bottom gives rise to a composition series, because index 2 subgroups are normal.

Exercise 3.4.5. Prove that subgroups and quotient groups of a solvable group are solvable.

Solution Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_s = G$ be a chain of subgroups of G such that G_{i+1}/G_i is abelian. (We call such chain a *solvable series* of G.)

(i) Let $H \leq G$. We have that $G_i \cap H \trianglelefteq G_{i+1} \cap H$ for all i (cf. [DF, Exercise 3.1.24]). By the Second Isomorphism Theorem ($A = G_{i+1} \cap H$ and $B = G_i$ with $A \leq N_G(B)$), $(G_{i+1} \cap H)/(G_i \cap H) \cong (G_{i+1} \cap H)(G_i)/G_i \leq G_{i+1}/G_i$ for all i. Thus $(G_{i+1} \cap H)/(G_i \cap H)$ is abelian. It follows that $1 = (G_0 \cap H) \trianglelefteq (G_1 \cap H) \trianglelefteq \ldots \trianglelefteq (G_s \cap H) = H$ is a solvable series of H.

(ii) Let $f : G \to K$ be an epimorphism. It suffices to show that K is solvable. (Why?) First, verify that $f(G_i) \trianglelefteq f(G_{i+1})$. Since G_{i+1}/G_i is abelian, for $a, b \in G_{i+1}$, we have $abG_i = baG_i$ hence $f(a)f(b)f(G_i) = f(abG_i) = f(baG_i) = f(b)f(a)f(G_i)$. Therefore $f(G_{i+1})/f(G_i)$ is abelian. It follows that $1 = f(G_0) \trianglelefteq f(G_1) \trianglelefteq \ldots \trianglelefteq f(G_s) = K$ is a solvable series of K.

Exercise 3.4.8. Let G be a *finite* group. Prove that the following are equivalent:

(i) G is solvable

(ii) G has a chain of subgroups: $1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is cyclic, $0 \leq i \leq s-1$

(iii) all composition factors of G are of prime order

(iv) G has a chain of subgroups: $1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N_i is abelian, $0 \leq i \leq t-1$.

Solution (i)⇒(iii) Lemma. If $N \trianglelefteq G$, G/N is abelian and $|G/N|$ is divisible by a prime p, then there is $N \trianglelefteq A \trianglelefteq G$ such that G/A is abelian and $|A/N| = p$. (*Proof*: Apply Cauchy's Theorem to $\overline{G} = G/N$ and use [DF, Theorem 20, page 99]. $G/A = (G/N)/(A/N)$ is abelian.)

Now let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_s = G$ be a solvable series of G. Apply Lemma to $G_{s-1} \trianglelefteq G_s$ getting $G_{s-1} \trianglelefteq H_1 \trianglelefteq G_s$, to $H_1 \trianglelefteq G_s$ getting $G_{s-1} \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq G_s$, and so on. Because G is finite, this process eventually stops. Repeating the same process for each pair $G_i \trianglelefteq G_{i+1}$, we obtain a desired composition series of G.

(iii)⇒(ii) Obvious. (ii)⇒(i) Clear. (iv)⇒(i) Clear.

(iii)⇒(iv) Let N_1 be a minimal non-trivial normal subgroup of G. We claim that N_1 is abelian. Choose a maximal normal subgroup N of N_1. By (iii) and maximality of N, $|N_1/N|$ is prime and thus N_1/N is abelian. It follows that $xyx^{-1}y^{-1} \in N$ for all $x, y \in N_1$. Since $N_1 \trianglelefteq G$, we have $gNg^{-1} \trianglelefteq gN_1g^{-1} = N_1$ for all $g \in G$. Because conjugation by g is an automorphism of G, gNg^{-1} is also a maximal normal subgroup of N_1. By the same reasoning as above, $xyx^{-1}y^{-1} \in gNg^{-1}$ for all $x, y \in N_1$ and $g \in G$, thus $xyx^{-1}y^{-1} \in \hat{N} := \cap_{g \in G}(gNg^{-1})$. Because $\hat{N} \trianglelefteq G$ and $\hat{N} \leq N_1$, we must have $\hat{N} = 1$ by minimality of N_1. Therefore, $xyx^{-1}y^{-1} = 1$ for all $x, y \in N_1$ and N_1 is abelian as claimed.

Now apply the same argument to $\overline{G}_1 := G/N_1$ to get a minimal non-trivial abelian normal subgroup $\overline{N}_2$ of $\overline{G}_1$. Then N_2 is a normal subgroup of G containing N_1 and N_2/N_1 is abelian. Repeating this process with $\overline{G}_i := G/N_i$, we eventually get a desired series $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots \trianglelefteq N_t = G$.

Exercise 3.5.9. Prove that the (unique) subgroup of order 4 in A_4 is normal and is isomorphic to V_4.

Solution Because the orders of elements of A_4 are $1, 2$ and 3, a subgroup N of order 4 in A_4 consists of elements of order 1 and 2 only. There are only four of them, thus $N = \{1, (12)(34), (13)(24), (14)(23)\}$. Conjugation does not change cycle types [DF, Proposition 10 and 11, pages 125-6] and $N \setminus \{1\}$ consists of all elements *in* A_4 of the same particular cycle type. It follows that N is normal in A_4. Obviously, N is isomorphic to V_4.

Exercise 3.5.10. Find a composition series for A_4. Deduce that A_4 is solvable.

Solution Let $N = \{1, (12)(34), (13)(24), (14)(23)\}$. Then $1 \trianglelefteq \langle (12)(34) \rangle \trianglelefteq N \trianglelefteq A_4$ is a composition series for A_4, because the orders of composition factors are 2 or 3. By Exercise 3.4.8.(iii), A_4 is solvable.