

250A Homework 8

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Exercise 10.1.8 (a). An element m of the R -module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted $\text{Tor}(M)$. Prove that if R is an integral domain then $\text{Tor}(M)$ is a submodule of M .

Solution We verify the submodule criterion [DF, p342]. (i) Since $1 \cdot 0 = 0$, we have $0 \in \text{Tor}(M) \neq \emptyset$. (ii) If $m_1, m_2 \in \text{Tor}(M)$ then there exist nonzero $r_1, r_2 \in R$ such that $r_1 m_1 = r_2 m_2 = 0$. Thus, for all $r \in R$, we have

$$\begin{aligned} r_1 r_2 (m_1 + r m_2) &= (r_1 r_2) m_1 + (r_1 r_2) r m_2 \\ &= r_2 (r_1 m_1) + r_1 r (r_2 m_2) \\ &= r_1 \cdot 0 + r_1 r \cdot 0 \\ &= 0, \end{aligned}$$

because R is commutative. Since $r_1 r_2$ is a nonzero element of R , this shows that $m_1 + r m_2 \in \text{Tor}(M)$.

Exercise 10.1.9. If N is a submodule of M , the *annihilator* $\text{Ann}(N)$ of N in R is defined to be

$$\text{Ann}(N) = \{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

Prove that the annihilator $\text{Ann}(N)$ is a 2-sided ideal of R .

Solution For any subset $S \subset M$ it is easy to verify that $\text{Ann}(S)$ is a left idea of R . Let N be a submodule of M . If $r \in R$ and $a \in \text{Ann}(N)$, then for every $n \in N$ we have $(ar)n = a(rn) = 0$ since $rn \in N$. Thus $ar \in \text{Ann}(N)$, and $\text{Ann}(N)$ is also a right ideal.

Exercise 10.1.11 (a). Let M be the \mathbb{Z} -module $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$. Find the annihilator of M in \mathbb{Z} .

Solution Let $l = \text{lcm}(24, 15, 50) = 600$. We claim that $\text{Ann}(M) = l\mathbb{Z} = \langle l \rangle$. It is clear that $\langle l \rangle \subset \text{Ann}(M)$. If $m \in \text{Ann}(M)$ then $m \in \text{Ann}(\mathbb{Z}/24\mathbb{Z}) \cap \text{Ann}(\mathbb{Z}/15\mathbb{Z}) \cap \text{Ann}(\mathbb{Z}/50\mathbb{Z})$, hence $24|m$, $15|m$ and $50|m$. Thus $l|m$, and $m \in \langle l \rangle$. Therefore, $\text{Ann}(M) \subset \langle l \rangle$.

Exercise 10.3.4. Prove that every finite abelian group is a torsion \mathbb{Z} -module. Give an example of an infinite abelian group that is a torsion \mathbb{Z} -module.

Solution Let A be a finite abelian group of order $|A| = n$. Given $a \in A$, n is divisible by the order of the subgroup $\langle a \rangle$ (Lagrange's theorem). Thus $na = 0$, hence $a \in \text{Tor}(A)$. Therefore, $A = \text{Tor}(A)$ is a torsion \mathbb{Z} -module.

\mathbb{Q}/\mathbb{Z} is an infinite abelian group that is a torsion \mathbb{Z} -module. Indeed, \mathbb{Q}/\mathbb{Z} has infinite order. For any given $\frac{m}{n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, we have $n(\frac{m}{n} + \mathbb{Z}) = m + \mathbb{Z} = 0 + \mathbb{Z}$. Thus $\frac{m}{n} + \mathbb{Z} \in \text{Tor}(\mathbb{Q}/\mathbb{Z})$.

Question. Consider an abelian group with generators a, b, c, d and relations

$$\begin{aligned} 6a + 4b + 12c + 18d &= 0, \\ 4a + 8c + 12d &= 0. \end{aligned}$$

Represent this group as a finite sum of cyclic groups.

Solution The following table shows that the given group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z} \oplus \mathbb{Z}$.

6	4	12	18	$6a + 4b + 12c + 18d = 0$
4	0	8	12	$4a + 8c + 12d = 0$
				a, b, c, d
2	4	4	6	$2a + 4b + 4c + 6d = 0$
4	0	8	12	$4a + 8c + 12d = 0$
				a, b, c, d
2	4	4	6	$2a + 4b + 4c + 6d = 0$
0	-8	0	0	$-8b = 0$
				a, b, c, d
2	0	4	6	$2(a + 2b) + 4c + 6d = 0$
0	-8	0	0	$-8b = 0$
				$a + 2b, b, c, d$
2	0	0	6	$2(a + 2b + 2c) + 6d = 0$
0	-8	0	0	$-8b = 0$
				$a + 2b + 2c, b, c, d$
2	0	0	0	$2(a + 2b + 2c + 3d) = 0$
0	-8	0	0	$-8b = 0$
				$a + 2b + 2c + 3d, b, c, d$