250B Homework 1

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Exercise 12.3.12. Determine the Jordan canonical form for the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution The characteristic polynomial of A is $(x-1)^4$. Thus A has only one eigenvalue 1 and the corresponding eigenspace $\ker(A-I) = \text{span}\{(1,0,0,0)^t\}$ is 1-dimensional. Therefore, the Jordan canonical form for A has only one Jordan block corresponding to the eigenvalue 1 and hence must be

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 12.3.18. Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$.

Solution The possible minimal polynomial of the linear transformation is $(x-2)^\alpha(x-3)^\beta$, where $1 \le \alpha \le 3$ and $1 \le \beta \le 2$. Thus all possible Jordan canonical forms are the followings (up to permutation of Jordan blocks).

$$\begin{pmatrix} 2&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 2&1&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 2&1&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix},$$
$$\begin{pmatrix} 2&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&3&1\\0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 2&1&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&3&1\\0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 2&1&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\0&0&0&3&1\\0&0&0&0&3 \end{pmatrix}.$$

Exercise 12.3.21. Show that if $A^2 = A$ then A is similar to a diagonal matrix which has only 0's and 1's along the diagonal.

Solution Because A satisfies $A^2 - A = 0$, its minimal polynomial divides $x^2 - x = x(x-1)$. Thus the eigenvalues of A are 0 or 1, and its minimal polynomial has no repeated roots. The conclusion now follows from [DF, Corollary 24 and 25, pages 493-494].

Exercise 12.3.22. Prove that an $n \times n$ matrix A with entries from $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over *any* field $\mathbb{F}$?

Solution Because the polynomial $x^3 - x = x(x-1)(x+1)$ has no repeated roots ***in*** $\mathbb{C}$, the same reasoning as in the solution of Exercise 12.3.21 shows that A is diagonalizable. If the entries of A are from the field $\mathbb{Z}_2$, however, then $x^3 - x = x(x-1)^2$ and it is possible that the minimal polynomial of A may have repeated roots, in which case A is not diagonalizable by [DF, Corollary 25, page 494].

Exercise 12.3.25. Determine the Jordan canonical form for the $n \times n$ matrix A over $\mathbb{Q}$ whose entries are all equal to 1.

Solution Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Q}^n$. One can check that A has n linearly independent eigenvectors: $e_1 + \cdots + e_n$ with eigenvalue n and $e_1 - e_k$ $(2 \le k \le n)$ with eigenvalue 0. From [DF, Exercise 11.2.8(b), page 423] we see that A is similar to the diagonal matrix

$$D(n, 0, \ldots, 0),$$

which is the Jordan canonical form for A by [DF, Corollary 24(1), page 493].