250B Homework 2

prepared by Jaejeong Lee

Question 1. Let us suppose that A is an antisymmetric real matrix (i.e. $A + A^t = 0$). Prove that e^A is an orthogonal matrix.

Solution One can verify that $(e^A)^t = e^{A^t}$. Because A and $-A$ commute, we have

$$e^A(e^A)^t = e^A e^{A^t} = e^A e^{-A} = e^{A-A} = e^0 = I.$$

Thus e^A is an orthogonal matrix.

Question 2. Prove that for a complex matrix A

$$\det e^A = e^{\mathrm{tr}A}.$$

Solution One can verify that if $X = PYP^{-1}$ for an invertible matrix P then $e^X = Pe^Y P^{-1}$, $\det X = \det Y$, and $\mathrm{tr}X = \mathrm{tr}Y$. Thus the given formula is satisfied by X if and only if it is satisfied by Y.

If $DN = ND$ one can check that if the given formula is satisfied by D and N then it is also satisfied by $D + N$.

The Jordan canonical form says that every complex matrix A is the sum of two commuting matrices D and N, with D diagonalizable and N nilpotent. Now it is easy to see that the given formula is satisfied by diagonal matrices D and by nilpotent matrices N. (*cf.* Exercise 12.3.31)

Question 3. Assume that V is a finite dimensional vector space and S is a subspace of V^*. Construct an isomorphism between S and the space dual to the quotient space $V/\mathrm{Ann}(S)$. (*Hint*. Construct a non-degenerate pairing between S and $V/\mathrm{Ann}(S)$.)

Solution Define a map $b : S \times [V/\mathrm{Ann}(S)] \to \mathbb{F}$ by $b(s, v + \mathrm{Ann}(S)) = s(v)$. One can check that b is well-defined and bilinear.

If $0 = b(s, v + \mathrm{Ann}(S)) = s(v)$ for all $v \in V$ then $s = 0$. If $0 = b(s, v + \mathrm{Ann}(S)) = s(v)$ for all $s \in S$ then $v \in \mathrm{Ann}(S)$ and hence $v + \mathrm{Ann}(S) = 0 + \mathrm{Ann}(S)$. Therefore, b is non-degenerate.

Now, define $\phi : S \to [V/\mathrm{Ann}(S)]^*$ by $\phi(s) = b(s, \cdot)$. Because b is a bilinear non-degenerate pairing, ϕ is an injective linear transformation. Because

$$\begin{aligned}\dim[V/\mathrm{Ann}(S)]^* &= \dim[V/\mathrm{Ann}(S)] \\ &= \dim V - \dim \mathrm{Ann}(S) = \dim S\end{aligned}$$

(see Exercise 11.3.3(f) below), we see that ϕ has to be an isomorphism.

Exercise 11.3.2. Let V be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5 with $1, x, x^2, \ldots, x^5$ as basis. Prove that the following are the elements of the dual space of V and express them as linear combinations of the dual basis:
(a) $E : V \to \mathbb{Q}$ defined by $E(p(x)) = p(3)$.
(c) $\phi : V \to \mathbb{Q}$ defined by $\phi(p(x)) = \int_0^1 t^2 p(t)dt$.
(d) $\phi : V \to \mathbb{Q}$ defined by $\phi(p(x)) = p'(5)$.

Solution In each case it is easy to see that the given map defines a linear functional on V.

(a) $E = 3^5(x^5)^* + 3^4(x^4)^* + 3^3(x^3)^* + 3^2(x^2)^* + 3x^* + 1^*$.

(c) $\phi = (1/8)(x^5)^* + (1/7)(x^4)^* + (1/6)(x^3)^* + (1/5)(x^2)^* + (1/4)x^* + (1/3)1^*$.

(d) $\phi = 5 \cdot 5^4(x^5)^* + 4 \cdot 5^3(x^4)^* + 3 \cdot 5^2(x^3)^* + 2 \cdot 5(x^2)^* + x^*$.

Exercise 11.3.3. Let S be any subset of V^* for some finite dimensional space V. Define

$$\mathrm{Ann}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}.$$

(a) Prove that $\mathrm{Ann}(S)$ is a subspace of V.
(b) Let W_1 and W_2 be subspaces of V^*. Prove that $\mathrm{Ann}(W_1 + W_2) = \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$ and $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.
(e) Assume V is finite dimensional with basis $v_1, \ldots, v_n$. Prove that if $S = \{v_1^*, \ldots, v_k^*\}$ for some $k \leq n$, then $\mathrm{Ann}(S)$ is the subspace spanned by $\{v_{k+1}, \ldots, v_n\}$.
(f) Assume V is finite dimensional. Prove that if W^* is any subspace of V^* then $\dim \mathrm{Ann}(W^*) = \dim V - \dim W^*$.

Solution (a) If $v, w \in \mathrm{Ann}(S)$ and $c \in \mathbb{F}$, then for all $f \in S$ we have $f(v+cw) = f(v)+cf(w) = 0+c\cdot 0 = 0$. Thus $v + cw \in \mathrm{Ann}(S)$.

(b) (i) $\text{Ann}(W_1+W_2) \subset \text{Ann}(W_1)\cap\text{Ann}(W_2)$ follows from $W_1 + W_2 \supset W_1$ and $W_1 + W_2 \supset W_2$. The reverse inclusion is clear.

(b) (ii) $\text{Ann}(W_1\cap W_2) \supset \text{Ann}(W_1)+\text{Ann}(W_2)$ follows from $W_1 \cap W_2 \subset W_1$ and $W_1 \cap W_2 \subset W_2$. The equality follows from dimension counting, for which we use the results in (b)(i) and (f). Let $n = \dim V$. We have

$$\begin{aligned}
&\dim[\text{Ann}(W_1) + \text{Ann}(W_2)] \\
&= \dim \text{Ann}(W_1) + \dim \text{Ann}(W_2) \\
&\qquad - \dim[\text{Ann}(W_1) \cap \text{Ann}(W_2)] \\
&= \dim \text{Ann}(W_1) + \dim \text{Ann}(W_2) \\
&\qquad - \dim[\text{Ann}(W_1 + W_2)] \\
&= (n - \dim W_1) + (n - \dim W_2) \\
&\qquad - (n - \dim(W_1 + W_2)) \\
&= (n - \dim W_1) + (n - \dim W_2) \\
&\qquad - n + [\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)] \\
&= n - \dim(W_1 \cap W_2) \\
&= \dim \text{Ann}(W_1 \cap W_2).
\end{aligned}$$

(e) Let $v = \sum_{i=1}^{n} c_i v_i$. Then $v \in \text{Ann}(S)$ if and only if $0 = v_j^*(v) = c_j$ for all $j = 1, \ldots, k$ if and only if $v \in \text{span}\{v_{k+1}, \ldots, v_n\}$.

(f) Choose a basis $\{f_1, \ldots, f_k\}$ for W^* and extend it to a basis $\{f_1, \ldots, f_n\}$ for V^*. Then $\{f_1^*, \ldots, f_n^*\}$ is a basis for V (assuming the natural isomorphism between V^{**} and V). Because $\{f_1^{**}, \ldots, f_k^{**}\} = \{f_1, \ldots, f_k\}$, we can apply (d) and (e) to see that $\text{Ann}(W^*)$ is the subspace spanned by $\{f_{k+1}^*, \ldots, f_n^*\}$. Therefore, $\dim \text{Ann}(W^*) = n - k = \dim V - \dim W^*$.