250B Homework 3

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Question 1. Diagonalize the following quadratic form over $\mathbb{R}$:

$$x_1^2 - x_2^2 + 2x_3^2 - 2x_1x_2 - 4x_1x_3.$$

Is this quadratic form nondegenerate? Is it positively definite?

Solution We have

$$\begin{aligned}
&x_1^2 - x_2^2 + 2x_3^2 - 2x_1x_2 - 4x_1x_3 \\
&= x_1^2 - 2(x_2 + 2x_3)x_1 - x_2^2 + 2x_3^2 \\
&= [x_1 - (x_2 + 2x_3)]^2 - (x_2 + 2x_3)^2 - x_2^2 + 2x_3^2 \\
&= [x_1 - (x_2 + 2x_3)]^2 - 2x_2^2 - 4x_2x_3 - 2x_3^2 \\
&= [x_1 - (x_2 + 2x_3)]^2 - 2(x_2 + x_3)^2.
\end{aligned}$$

Thus if we change the coordinates by

$$y_1 = x_1 - x_2 - 2x_3, \;\; y_2 = x_2 + x_3, \;\; y_3 = x_3$$

then the given quadratic form diagonalizes to

$$y_1^2 - 2y_2^2.$$

The quadratic form is degenerate (the coefficient of y_3 is 0) and is not positive definite (the coefficient of y_2 is negative).

Alternative solution: The matrix of the given quadratic form is

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -1 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

We can diagonalize the symmetric matrix A by performing a column operation E_1, a row operation E_1^t, a column operation E_2, a row operation E_2^t, and so on. That is,

$$D = P^tAP = E_k^t \cdots E_1^t A E_1 \cdots E_k.$$

Then $P = E_1 \cdots E_k$ is obtained by performing the same column operations on I. We thus have

$$\begin{aligned}
(A|I) &= \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & 0 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).
\end{aligned}$$

Under the coordinate change $x = Py$ the given quadratic form diagonalizes to

$$x^tAx = y^tP^tAPy = y_1^2 - 2y_2^2.$$

The quadratic form is degenerate (D has a 0 in the diagonal) and is not positive definite (D has a negative number in the diagonal).

Question 2. For every matrix $A \in GL_n$ we define a linear operator acting on the vector space Mat_n of $n \times n$ matrices by means of one of the following formulas:

$$\begin{aligned}
\phi_A(X) &= AX, \\
\psi_A(X) &= AXA, \\
\lambda_A(X) &= AXA^t, \\
\mu_A(X) &= AXA^{-1}.
\end{aligned}$$

Single out the formulas defining a linear representation of the group GL_n.

Solution It is clear that each formula defines an invertible linear operator on Mat_n. For example, we check that $\phi_A(X + cY) = \phi_A(X) + c\phi_A(Y)$ and ϕ_A is invertible.

To see if ϕ defines a linear representation of GL_n, we need to check that the map $\phi : GL_n \to GL(Mat_n)$ given by $A \mapsto \phi_A$ is a homomorphism. Indeed, for any $A, B \in GL_n$ and $X \in Mat_n$, we have

$$\phi_{AB}(X) = ABX = A(BX) = (\phi_A \circ \phi_B)(X)$$

and hence $\phi_{AB} = \phi_A \circ \phi_B$. Similarly, we check that

$$\begin{aligned}\lambda_{AB}(X) &= (AB)X(AB)^t \\ &= A(BXB^t)A^t = (\lambda_A \circ \lambda_B)(X), \\ \mu_{AB}(X) &= (AB)X(AB)^{-1} \\ &= A(BXB^{-1})A^{-1} = (\mu_A \circ \mu_B)(X).\end{aligned}$$

Thus, λ and μ also define linear representations of GL_n.

However, because $\psi_{AB}(X) = (AB)X(AB)$ and $(\psi_A \circ \psi_B)(X) = A(BXB)A$, and in general $AB \neq BA$ unless $n = 1$, we see that ψ does not define a linear representation of GL_n for $n > 1$.

Question 3. Let us denote the subspace of Mat_n consisting of symmetric matrices by S and the subspace consisting of matrices having zero trace (traceless matrices) by T. Are these subspaces invariant with respect to representations of the group GL_n described in the problem 2? Are these representations of the group GL_n reducible?

Solution We claim that S is invariant with respect to λ and T is invariant with respect to μ. Indeed, for any $X \in S$, $Y \in T$ and $A \in GL_n$, we have

$$\begin{aligned}(\lambda_A(X))^t &= (AXA^t)^t = AX^tA^t = AXA^t = \lambda_A(X); \\ \text{tr}(\mu_A(Y)) &= \text{tr}(AYA^{-1}) = \text{tr}(AA^{-1}Y) = \text{tr}(Y) = 0,\end{aligned}$$

because $X^t = X$ and $\text{tr}(Y) = 0$. Therefore, λ and μ are reducible representations.

Let M_k be the subspace of Mat_n consisting of matrices X such that $X_{ij} = 0$ for $j \neq k$, that is, all columns of X are zero columns except possibly the k-th column. Then it is clear that for any $X \in M_k$ and $A \in GL_n$ we have $\phi_A(X) = AX \in M_k$. Thus M_k is invariant with respect to ϕ, and this shows that ϕ is also a reducible representation.

On the other hand, one can find counterexamples which show that (i) neither S nor T is invariant with respect to ϕ, (ii) S is not invariant with respect to μ, and (iii) T is not invariant with respect to λ:

(i)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, AX = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin S;$$
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, AX = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin T.$$

(ii)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, AXA^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin S.$$

(iii)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, AXA^t = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin T.$$

Question: Is it true that the subrepresentations ϕ_{M_k}, λ_S, and μ_T are irreducible?