Homework 5

Problem 1

During lecture it was proven that that $\mathbb{Z}_n \otimes Y$ is the unique vector space for which we can make the identification $B(\mathbb{Z}_n \times Y, Z) = L(\mathbb{Z}_n \otimes Y, Z)$. Hence it suffices to exhibit a bijection between $B(\mathbb{Z}_n \times Y, Z)$ and $L(Y/nY, Z)$ since $L(Y/nY, Z)$ can be identified with the set of homomorphisms ϕ which obey $n\phi(y) = 0$ for all y. To this end, we construct an injection from $L(Y/nY, Z)$ to $B(\mathbb{Z}_n \times Y, Z)$ and vice-versa.

Define $f : L(Y/nY, Z) \to B(\mathbb{Z}_n \times Y, Z)$ by $f(\phi) = \beta_\phi$, where $\beta_\phi(k, y) = k\phi(y + nY)$ for $0 \le k \le n-1$ and all $y \in Y$. Because ϕ is linear, β_ϕ is linear in both arguments, which makes β_ϕ a bihomomorphism. Moreover, if $\beta_\phi(k, y) = 0$ for all $1 \le k \le n$ and $y \in Y$, then in particular $\beta_\phi(1, y) = 0$ for all $y \in Y$, i.e., $\phi(y + nY) = 0$ for all $y \in Y$. Thus f is injective.

Likewise, define $g : B(\mathbb{Z}_n \times Y, Z) \to L(Y/nY, Z)$ by $g(\beta) = \phi_\beta$, where $\phi_\beta(y + nY) = \beta(1, y)$. First we verify that the ϕ_β are well-defined. For $y_0, y \in Y$, we have:

$$\begin{aligned}
\phi_\beta(y_0 + ny) &= \beta(1, y_0 + ny) = \beta(1, y_0) + n\beta(1, y) \\
&= \beta(1, y_0) + \beta(n, y) \\
&= \beta(1, y_0) + \beta(0, y) \\
&= \beta(1, y_0) \\
&= \phi_\beta(y_0)
\end{aligned}$$

From this we conclude that the ϕ_β are well-defined. Moreover each ϕ_β is linear since β is linear in both arguments. Finally suppose that $\phi_\beta(y + nY) = 0$ for all $y \in Y$. Then $\beta(1, y) = 0$ for all $y \in Y$. Thus $\beta(k, y) = k\beta(1, y) = 0$ for all $0 \le k \le n-1$ and $y \in Y$ and we conclude that g is injective.

By the Schroeder-Bernstein Theorem and our work in the preceding paragraphs, there exists a bijection h between $L(Y/nY, Z)$ and $B(\mathbb{Z}_n \times Y, Z)$. Consequently, we conclude that $\mathbb{Z}_n \otimes Y = Y/nY$ (as we discussed above).

Problem 2

Let $\vec{x} = (x, y, z)$ and let us denote a polynomial in three variables by $p(\vec{x})$. Then the representation of GL(3) is given by the map $A \mapsto \varphi_A$, where $\varphi_A[p(\vec{x})] = p(A\vec{x})$.

a) In order for Δ to be an endomorphism of E, it must respect the action of GL(3) on E, i.e., we must have $\Delta\varphi_A = \varphi_A\Delta$ for all $A \in \mathrm{GL}(3)$. However, if we take:

$$p(\vec{x}) = x^2, \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

then $\varphi_A \Delta[p(\vec{x})] = \varphi_A(2) = 2$ whereas $\Delta \varphi_A[p(\vec{x})] = \Delta[(x+y)^2] = 4$.

Let us now restrict our attention to the subgroup $O(3)$ of GL(3). Clearly Δ is linear since differentiation is a linear operator. It remains to show that Δ respects the action of $O(3)$ on E, i.e., that $\Delta\varphi_A(p) = \varphi_A(\Delta p)$ for all $A \in O(3)$. It suffices to prove the claim for monic monomials, i.e., for polynomials of the form $p(\vec{x}) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. Then the same argument extends by linearity to general polynomials p.

Let $p(\vec{x}) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ and $A = (a_{ij}) \in O(3)$, and for $1 \le i \le 3$ define:

$$M_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \tag{2}$$

Then using the product rule for differentiation, we have:

$$\varphi_A[\Delta(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3})] = \varphi_A \left[\sum_{i,j,k} (\Delta x_i^{\alpha_i}) x_j^{\alpha_j} x_k^{\alpha_k} \right] = \sum_{i,j,k} \alpha_i(\alpha_i - 1) M_i^{\alpha_i - 2} M_j^{\alpha_j} M_k^{\alpha_k} \tag{3}$$

Likewise:

$$\begin{aligned}
\Delta[\varphi_A(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3})] &= \Delta(M_1^{\alpha_1} M_2^{\alpha_2} M_3^{\alpha_3}) & (4)\\
&= \sum_{i,j,k} (\Delta M_i^{\alpha_i}) M_j^{\alpha_j} M_k^{\alpha_k} + 2 \left\{ \begin{matrix} \sum_{i,j,k} M_i^{\alpha_i} (\partial_{x_1} M_j^{\alpha_j})(\partial_{x_1} M_k^{\alpha_k}) \\ \sum_{i,j,k} M_i^{\alpha_i} (\partial_{x_2} M_j^{\alpha_j})(\partial_{x_2} M_k^{\alpha_k}) \\ \sum_{i,j,k} M_i^{\alpha_i} (\partial_{x_3} M_j^{\alpha_j})(\partial_{x_3} M_k^{\alpha_k}) \end{matrix} \right\} & (5)\\
&= \sum_{i,j,k} \alpha_i(\alpha_i - 1) M_i^{\alpha_i - 2} (a_{i1}^2 + a_{i2}^2 + a_{i3}^2) M_j^{\alpha_j} M_k^{\alpha_k} & (6)\\
&\qquad + 2 \left\{ \begin{matrix} \sum_{i,j,k} \alpha_j \alpha_k (a_{j1} a_{k1}) M_i^{\alpha_i} M_j^{\alpha_j - 1} M_k^{\alpha_k - 1} \\ \sum_{i,j,k} \alpha_j \alpha_k (a_{j2} a_{k2}) M_i^{\alpha_i} M_j^{\alpha_j - 1} M_k^{\alpha_k - 1} \\ \sum_{i,j,k} \alpha_j \alpha_k (a_{j3} a_{k3}) M_i^{\alpha_i} M_j^{\alpha_j - 1} M_k^{\alpha_k - 1} \end{matrix} \right\} & (7)\\
&= \sum_{i,j,k} \alpha_i(\alpha_i - 1) M_i^{\alpha_i - 2} (a_{i1}^2 + a_{i2}^2 + a_{i3}^2) M_j^{\alpha_j} M_k^{\alpha_k} & (8)\\
&\qquad + 2 \sum_{i,j,k} \alpha_j \alpha_k (\textstyle\sum_{\ell=1}^3 a_{j\ell} a_{k\ell}) M_i^{\alpha_i} M_j^{\alpha_j - 1} M_k^{\alpha_k - 1} & (9)
\end{aligned}$$

But since A is orthogonal and $j \neq k$ in the above summations, we have:

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1 \quad \text{and} \quad \sum_{\ell=1}^{3} a_{j\ell} a_{k\ell} = 0 \tag{10}$$

Putting the equations of (10) into (8) and (9) respectively, we see that:

$$\Delta[\varphi_A(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3})] = \sum_{i,j,k} \alpha_i(\alpha_i - 1) M_i^{\alpha_i - 2} M_j^{\alpha_j} M_k^{\alpha_k} \tag{11}$$

Comparing (3) and (11), it follows that $\varphi_A(\Delta p) = \Delta\varphi_A(p)$ for all monic monomials $p \in E$ and this argument extends by linearity to all polynomials $p \in E$. Thus we conclude that Δ is an endomorphism of E regarded as a representation of $O(3)$.

b) It is clear that the set of harmonic polynomials is a linear subspace of E. In part (a), we also proved that $\Delta\varphi_A = \varphi_A\Delta$ for all $A \in O(3)$. Hence if p is a harmonic polynomial, then $\Delta\varphi_A(p) = \varphi_A(\Delta p) = \varphi_A(0) = 0$, and therefore $\varphi_A(p)$ is also a harmonic polynomial for any $A \in O(3)$. Thus we conclude that the harmonic polynomials form an invariant subspace with respect to $O(3)$.

c) Let $x = (x_1, x_2, x_3)$, where x_1, x_2, x_3 are indeterminates. It is easy to see that any homogeneous polynomial $p(x)$ of degree 2 can be written in the form:

$$p(x) = x^t Ax, \quad \text{where } A \text{ is symmetric} \tag{12}$$

In fact, if $p(x) = \sum_{i\leq j} a_{ij}x_i x_j$, then one can verify that:

$$A_{ij} = A_{ji} = \begin{cases} a_{ii}, & \text{if } i = j \\ \frac{1}{2}a_{ij}, & \text{if } i \neq j \end{cases} \tag{13}$$

Conversely, any polynomial of the form (12) is homogeneous and of degree 2. Thus p is a homogeneous polynomial of degree 2 iff $p(x) = x^t Ax$ for some symmetric matrix A.

Now define V to be the space of homogeneous polynomials of degree 2 and choose any $p(x) = x^t Ax \in V$ with A symmetric. For every invertible matrix Q, we have:

$$\varphi_Q[p(x)] = p(Qx) = (Qx)^t A(Qx) = x^t(Q^t AQ)x \tag{14}$$

Clearly $Q^t AQ$ is also symmetric. Hence $\varphi_Q(p) \in V$ for every $Q \in \mathrm{GL}(3)$ and we conclude that the homogeneous polynomials of degree 2 are GL(3)-invariant.

Moreover this representation of GL(3) in V is irreducible. Let W be any invariant subspace of V containing a nonzero homogeneous polynomial $p(x) = x^t Ax$ of degree 2. Using the procedure described in Problem 1 of Homework 3, we can diagonalize p by means of invertible matrices. More specifically, A is congruent to a diagonal matrix. Moreover, we can also assume that A is congruent to a matrix with only 0s and $\pm$1s on its main diagonal. For example, if:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

and:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

then:

$$\varphi_Q(p(x)) = p(Qx) = x^t(Q^t AQ)x = x^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \tag{17}$$

Hence W, being invariant under the action of GL(3), contains a polynomial $p(x) = xAx^t$ such that A is diagonal with only 0s and $\pm$1s on its main diagonal. By hypothesis this A is nonzero. Hence without loss of generality we can assume it contains at least one nonzero entry (either 1 or -1) on its main diagonal. Using another invertible matrix on p we can move this entry to the $(1,1)$ position in the matrix. Now if we take:

$$Q = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

then:

$$a_{11}\left[\varphi_Q(p(x)) - p(x)\right] = a_{11}\left[x^t(Q^tAQ - A)x\right] = x^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \tag{19}$$

Hence W contains a polynomial of the form (19) since W is invariant under the action of GL(3) and is closed under addition and scalar multiplication. Now it is easy to see that using the polynomial in (19), we can obtain any polynomial of the form (12), first by acting on (19) with invertible matrices in order to move the 1 to any other position in the matrix, and then by taking arbitrary linear combinations to get the desired values in each entry. Thus $W = V$, and we conclude that this representation of GL(3) in V is irreducible.

On the other hand, the representation of $O(3)$ in V is reducible. In part (b) we proved that the harmonic polynomials formed a proper nontrivial subrepresentation of V. It follows immediately that the representation of $O(3)$ in V is reducible.

Problem 3

Let T be the space of traceless matrices and D be the space of matrices which are scalar multiples of the identity matrix. First we show that T and D are invariant under f for any endomorphism f of $\mathrm{M}_{n\times n}(\mathbb{C})$. Then we use this to characterize all endomorphisms f of $\mathrm{M}_{n\times n}(\mathbb{C})$.

Consider $f|_T : T \to \mathrm{M}_{n\times n}(\mathbb{C})$ and $f|_D : D \to \mathrm{M}_{n\times n}(\mathbb{C})$, the restrictions of f to T and D respectively, and $p_T : \mathrm{M}_{n\times n}(\mathbb{C}) \to T$ and $p_D : \mathrm{M}_{n\times n}(\mathbb{C}) \to D$, the projections of $\mathrm{M}_{n\times n}(\mathbb{C})$ onto T and D respectively defined by $p_T(X) = X_T$ and $p_D(X) = X_D$ where $X = X_T + X_D$ is the decomposition of X into its components in the direct sum $\mathrm{M}_{n\times n}(\mathbb{C}) = T \oplus D$. It is straightforward to verify that $(p_D \circ f|_T) : T \to D$ is a homomorphism from T to D. But T is irreducible and T and D are clearly not isomorphic (their dimensions do not match). By Schur's Lemma we conclude that $p_D \circ f|_T$ is the zero map, and thus f maps T into $\ker p_D$, which is T. Hence T is invariant under f. Using an identical proof, one can also prove that $(p_T \circ f|_D) : D \to T$ is the zero map, from which it follows that D is also invariant under f, as desired.

Finally, let f be any endomorphism of $\mathrm{M}_{n\times n}(\mathbb{C})$. It follows from our work above that T and D are invariant with respect to f. Moreover we proved in Homework 4 that T and D are irreducible. Hence by Schur's Lemma it follows that f restricted to T or D is precisely scalar multiplication. More specifically, for any matrix $X = X_T + X_D \in \mathrm{M}_{n\times n}(\mathbb{C})$ with $X_T \in T$ and $X_D \in D$, we have $f(X) = \lambda X_T + \mu X_D$ for some scalars $\lambda, \mu \in \mathbb{C}$. One can easily verify that any function satisfying this condition is also an endomorphism. Hence any endomorphism of this representation is completely determined by the scalars λ, μ. In this way we identify the endomorphism ring of the representation with the ring $\mathbb{C}^2$ under the operations of pointwise addition and multiplication.

Problem 4

If A is diagonalizable, then there exists a basis $\{x_1, \ldots, x_n\}$ of $\mathbb{C}^n$ which consists of eigenvectors of A. One can easily verify that if x_i is an eigenvector of A corresponding to an eigenvalue of λ_i, then x_i is also an eigenvector of e^{tA} corresponding to eigenvalue e^{λ_i}. Thus $\{x_1, \ldots, x_n\}$ is also a basis of eigenvectors of e^{tA} for all $t \in \mathbb{R}$. Hence we can write $\mathbb{C}^n = X_1 \oplus \cdots \oplus X_n$ where $X_i = \text{span}\{x_i\}$ for all i. Clearly each X_i is invariant with respect to e^{tA} according to our previous argument and each X_i is irreducible since each one is one-dimensional. Thus we conclude that the representation of $\mathbb{R}$ in $\mathbb{C}^n$ is completely reducible.

Conversely, suppose that the representation is completely reducible but that A is not diagonalizable. In the notes on linear representations, there is a statement which says that an irreducible complex representation of an abelian group is one-dimesional. Hence we can write $\mathbb{C}^n = X_1 \oplus \cdots \oplus X_n$, where each X_i is a one-dimensional irreducible subspace of $\mathbb{C}^n$ and invariant with respect to e^{tA} for all $t \in \mathbb{R}$. In particular, by taking $t = 1$, we see that each X_i is invariant with respect to e^A. Hence $\mathbb{C}^n$ has a basis of eigenvectors of e^A, which means that e^A is diagonalizable. However, A itself is not diagonalizable, and so the Jordan canonical form of A contains at least one 1 on its superdiagonal. We illustrate the case where A is a 2×2 matrix. It is easy to see that the proof generalizes to $n \times n$ matrices. If A is a 2×2 nondiagonalizable matrix, then its Jordan canonical form $J = Q^{-1}AQ$ consists of a single Jordan block:

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{20}$$

From our work on the final exam last quarter, we can find an expression for e^A:

$$e^A = Qe^JQ^{-1} = Q \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix} Q^{-1} \tag{21}$$

From this it is clear that the characteristic polynomial of e^A is $p(t) = (t - e^\lambda)^2$. Moreover the minimal polynomial $m(t)$ of e^A divides $p(t)$ but one can verify that $e^A - e^\lambda I \neq 0$. Hence $m(t) = (t - e^\lambda)^2$. By Corollary 25 of [DF, p. 494], it follows that e^A is not diagonalizable, which is a contradiction. This method readily generalizes to $n \times n$ matrices, where the proof is identical except for the number of Jordan blocks in J. Thus we conclude that A is diagonalizable.