

## 250B Homework 6

**Question 1.** The character table of finite group has the form

$$\begin{array}{c|ccccc} \text{size} & 1 & 6 & 8 & 6 & 3 \\ \chi_1 & 1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & -1 & 1 & -1 & 1 \\ \chi_3 & 2 & 0 & -1 & 0 & 2 \\ \chi_4 & 3 & 1 & 0 & -1 & -1 \\ \chi_5 & ? & ? & ? & ? & ? \end{array}$$

(The first line gives the sizes of conjugacy classes.)

- Find the character  $\chi_5$ .
- What are the dimensions of irreps?
- Decompose into a direct sum of irreps the tensor square of representation with the character  $\chi_4$ .
- Decompose into a direct sum of irreps the space of linear maps  $L(V_3, V_4)$ , where  $V_i$  stands for a space of irrep with character  $\chi_i$ .

**Solution** Note that the order of the group is

$$|G| = 1 + 6 + 8 + 6 + 3 = 24.$$

Denote  $\rho_i$  the irrep with character  $\chi_i$ .

(b) The dimension of  $\rho_i$  is  $\chi_i(1)$ . The dimensions of irreps satisfy

$$\begin{aligned} |G| &= \chi_1(1)^2 + \chi_2(1)^2 + \chi_3(1)^2 + \chi_4(1)^2 + \chi_5(1)^2 \\ &= 1 + 1 + 4 + 9 + \chi_5(1)^2. \end{aligned}$$

Therefore,  $\chi_5(1) = 3$  and the dimensions of irreps  $\rho_i$  are

$$1, 1, 2, 3, 3.$$

(a) The character  $\chi_{\text{reg}}$  of the regular representation of  $G$  has values  $(|G|, 0, 0, 0, 0)$  and satisfies

$$\chi_{\text{reg}} = 1 \cdot \chi_1 + 1 \cdot \chi_2 + 2 \cdot \chi_3 + 3 \cdot \chi_4 + 3 \cdot \chi_5.$$

We calculate

$$\begin{aligned} 3 \cdot \chi_5 &= \chi_{\text{reg}} - 1 \cdot \chi_1 - 1 \cdot \chi_2 - 2 \cdot \chi_3 - 3 \cdot \chi_4 \\ &= (24, 0, 0, 0, 0) - (1, 1, 1, 1, 1) - (1, -1, 1, -1, 1) \\ &\quad - 2(2, 0, -1, 0, 2) - 3(3, 1, 0, -1, -1) \\ &= (9, -3, 0, 3, -3). \end{aligned}$$

Thus we have  $\chi_5 = (3, -1, 0, 1, -1)$ .

(c) Because  $\chi_{\rho_4 \otimes \rho_4} = \chi_4 \cdot \chi_4 = (9, 1, 0, 1, 1)$ , we have

$$\begin{aligned} &\langle \chi_{\rho_3 \otimes \rho_3}, \chi_1 \rangle \\ &= \frac{1}{24} (1 \cdot 9 \cdot 1 + 6 \cdot 1 \cdot 1 + 8 \cdot 0 \cdot 1 + 6 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1) \\ &= 1 \end{aligned}$$

and similarly we get

$$\begin{aligned} \langle \chi_{\rho_4 \otimes \rho_4}, \chi_2 \rangle &= 0, \\ \langle \chi_{\rho_4 \otimes \rho_4}, \chi_3 \rangle &= 1, \\ \langle \chi_{\rho_4 \otimes \rho_4}, \chi_4 \rangle &= 1, \\ \langle \chi_{\rho_4 \otimes \rho_4}, \chi_5 \rangle &= 1. \end{aligned}$$

Therefore, we conclude that

$$\rho_4 \otimes \rho_4 \cong \rho_1 \oplus \rho_3 \oplus \rho_4 \oplus \rho_5.$$

(d) Because  $L(V_3, V_4) \cong V_3^* \otimes V_4$  and  $\rho_3$  is real, the given representation is equivalent to  $\rho_3^* \otimes \rho_4 = \rho_3 \otimes \rho_4$  and hence  $\chi_{L(V_3, V_4)} = \chi_3 \cdot \chi_4 = (6, 0, 0, 0, -2)$ . We calculate as in (c) to get

$$\begin{aligned} \langle \chi_{L(V_3, V_4)}, \chi_1 \rangle &= 0 \\ \langle \chi_{L(V_3, V_4)}, \chi_2 \rangle &= 0 \\ \langle \chi_{L(V_3, V_4)}, \chi_3 \rangle &= 0 \\ \langle \chi_{L(V_3, V_4)}, \chi_4 \rangle &= 1 \\ \langle \chi_{L(V_3, V_4)}, \chi_5 \rangle &= 1 \end{aligned}$$

Therefore,  $\rho_{L(V_3, V_4)} \cong \rho_4 \oplus \rho_5$ .

**Question 2.** Write down the character table of the symmetry group of regular pentagon (of dihedral group  $D_{10}$ .) For every representation find the dual representation.

**Solution** The class equation for the group  $D_{10}$  is

$$10 = 1 + 2 + 2 + 5,$$

where the conjugacy classes are

$$\begin{aligned} &\{\text{id}\}, \\ &\{\text{rotations by } \pm 2\pi/5\}, \\ &\{\text{rotations by } \pm 4\pi/5\}, \\ &\{\text{reflections}\}. \end{aligned}$$

So there are four different irreps and their dimensions should satisfy the relation

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 10.$$

The only solution of this equation is 1, 1, 2, 2. One 1-dimensional representation is trivial. The other 1-dimensional representation is the determinant: 1 for orientation preserving transformations and  $-1$  for orientation reversing ones ( $\chi_2(g) = -1$  for reflections and  $\chi_2(g) = 1$  for the rest of the group). Also, there is a natural 2-dimensional representation (in the plane of the pentagon). The trace of the rotation by the angle  $\varphi$

$$\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

is  $2 \cos \varphi$ . The trace of a reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is 0. Thus we know the characters of three representations:

$$\begin{pmatrix} \text{size} & 1 & 2 & 2 & 5 \\ \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & 1 & -1 \\ \chi_3 & 2 & 2 \cos \frac{2\pi}{5} & 2 \cos \frac{4\pi}{5} & 0 \\ \chi_4 & 2 & ? & ? & ? \end{pmatrix}$$

The remaining character can be found as in Question 1(a). Namely,  $2\chi_4 = (10, 0, 0, 0) - \chi_1 - \chi_2 - 2\chi_3$ , which gives  $\chi_4 = (2, 1 - 2 \cos \frac{2\pi}{5}, 1 - 2 \cos \frac{4\pi}{5}, 0)$ . From trigonometric formulas (or from the orthogonality of  $\chi_1$  and  $\chi_3$ ) one can deduce that  $1 - 2 \cos \frac{2\pi}{5} = 2 \cos \frac{4\pi}{5}$ . Thus the final table of characters is

$$\begin{pmatrix} \text{size} & 1 & 2 & 2 & 5 \\ \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & 1 & -1 \\ \chi_3 & 2 & 2 \cos \frac{2\pi}{5} & 2 \cos \frac{4\pi}{5} & 0 \\ \chi_4 & 2 & 2 \cos \frac{4\pi}{5} & 2 \cos \frac{2\pi}{5} & 0 \end{pmatrix}$$

Since all the characters are real (or, in other words, since each element of the group  $D_{10}$  is conjugated to its inverse), every irreducible representation of  $D_{10}$  is isomorphic to its dual representation.

**Question 3.** One says that a representation is *isotypic* if it can be represented as a direct sum of isomorphic irreps. If  $S$  is an irrep then an isotypic subrepresentation  $W$  of representation  $V$  is called an *isotypic component* of  $V$  if every subrepresentation of  $V$  isomorphic to  $S$  is a subrepresentation of  $W$ .

(a) Prove that every completely reducible representation has a unique decomposition into direct sum of isotypic components.

(b) Check that the group of automorphisms of isotypic representation is isomorphic to  $GL_n$ .

(c) Describe the group of automorphisms of direct sum of two isotypic representations.

**Solution** (a) Let  $V = V_1 \oplus \dots \oplus V_q = V'_1 \oplus \dots \oplus V'_q$  be two decompositions of  $V$  into irreps. Let  $W$  be the sum of all summands  $V_i$  which are isomorphic to  $S$  and  $Z$  be the sum of the remaining summands; thus  $V = W \oplus Z$  where  $W$  is the isotypic component of type  $S$ . Let  $V = W' \oplus Z'$  be a similar decomposition corresponding to the second decomposition of  $V$  into irreps. Consider the restriction of the projection  $V \rightarrow Z'$  to  $W$ . Since  $W$  and  $Z'$  have no isomorphic components, this restriction is 0 by the Schur's Lemma, that is,  $W$  is contained in the kernel  $W'$  of the projection  $V \rightarrow Z'$ , hence  $W \subset W'$ . Similarly,  $W' \subset W$ . Therefore,  $W = W'$ . The same is true for all isotypic components of  $V$ .

(b) Let  $W = S_1 \oplus \dots \oplus S_n$  where all  $S_i$  are irreps isomorphic to  $S$ . A linear map  $\varphi: W \rightarrow W$  is determined by linear maps  $\varphi_{ij}: S = S_i \subset W \xrightarrow{\text{pr.}} S_j = S$ . If  $\varphi$  is a homomorphism then all  $\varphi_{ij}$  are homomorphisms and, by the Schur's Lemma, each of them is a multiplication by a constant  $a_{ij} \in \mathbb{C}$ . Thus  $\varphi$  is determined by a matrix  $\{a_{ij}\}$  and  $\varphi$  is an isomorphism (automorphism of  $W$ ) if and only if this matrix is invertible. Hence, the group of automorphisms is  $GL_n$ .

(c) Let  $V = W \oplus Z$  where  $W$  and  $Z$  are isotypic

$$W = S_1 \oplus \dots \oplus S_m, \quad Z = T_1 \oplus \dots \oplus T_n$$

where  $S_i \cong S$ ,  $T_j \cong T$  for irreps  $S$  and  $T$ . If  $S$  and  $T$  are not isomorphic, then any automorphism of  $V$  takes  $W$  into  $W$  and  $Z$  into  $Z$ , so it splits into automorphisms of  $W$  and  $Z$ . By (b), the group of automorphisms in this case is  $GL_m \times GL_n$ . If  $S \cong T$ , then, again by (b), the group is  $GL_{m+n}$ .